Доклады Болгарской академии наук
Comptes rendus de l'Académie bulgare des Sciences
Tome 34, N° 11, 1981

MATHÉMATIQUES
Analyse fonctionnelle

DIFFERENTIABILITY OF CONVEX MAPPINGS AND GENERALIZED MONOTONE MAPPINGS

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(Submitted by Academician Bl. Sendov on June 11, 1981)

The main purpose of this note is to report on some conditions under which a given convex mapping $F: X \to Y$ (with values in the normed lattice Y and defined in the Banach space X) is Gâteaux or Frechet differentiable in the "majority" of the points of its domain of continuity, i. e. at the points of some dense G_δ subset of dom (F). This problem is well known in the particular case when Y is the real line [9-11]. Spaces named Asplund spaces (weak Asplund spaces) are just those Banach spaces X in which every continuous convex real-valued function $f: X \to R$ is Frechet (Gâteaux) differentiable at the points of some dense G_δ subset of X.

The convex mappings have been studied in [1,3,7,10,13].

Theorem 1. Let Y be an order complete Banach lattice. The following condition are equivalent:

a) Every continuous convex mapping $F: R^n \to Y$ is Frechet differentiable at the points of a dense G_δ subset of the Euclidean space R^n.

b) Y has weak compact intervals, i. e. the sets $[y_1, y_2]=\{y \in Y: y_1 \leq y \leq y_2\}$ are compact in $(Y, w(Y, Y^*))$.

Remark. Let us note that a continuous convex mapping $P: l_2 \to l_2$ exists, which is nowhere Frechet differentiable (see Example 2).

Theorem 2. Let X be a weak Asplund space and Y be an order complete separable normed lattice. Then every continuous convex mapping $F: X \to Y$ is Gâteaux differentiable at the points of some dense G_δ subset of X.

These theorems are obtained as corollaries of some results for generalized monotone mappings.

Definition. The multivalued mapping $T: X \to L(X, Y)$ is called generalized monotone mapping (GMM), if

$$(A_1-A_2)(x_1-x_2) \geq 0,$$

whenever $x_i \in X$, $A_i \in Tx_i$, $i=1, 2$ (see [7]). $L(X, Y)$ is the space of all bounded linear mappings from X into Y.

In the special case when $Y=R$, the definition coincides with the well known definition of monotone mappings [8,4]. The subdifferential mapping $\partial_F(x) =\{A \in L(X, Y): Ax-Az \leq Fx-Fz$ for all $z \in X\}$ of a convex mapping $F: X \to Y$ is a GMM. The multivalued mapping $T: X \to L(X, Y)$ is a GMM iff the multivalued mappings $(y^* \circ T): X \to X^*$, $(y^* \circ T)x=\{y^* \circ A: A \in Tx\}$, are monotone for every $y^* \in Y_+^*$ (Y_+^* is the conjugate cone of $Y_+=\{y \in Y: y \geq 0\}$).

Theorem 3. Suppose X is a Banach space, Y is a normed lattice and $T: X \to L(X, Y)$ is a GMM with non empty images. Then T is locally bounded at every point of X, i. e. for $x \in X$ there exists a neighbourhood V of x, such that the set $T(V) = \cup\{Tx : x \in V\}$ is a bounded subset of $L(X, Y)$.

In the case of $Y = R$ this theorem was proved by Rockafellar [12].

Corollary. If X is a Banach space, Y is an order complete normed lattice and $F: X \to Y$ is a continuous convex mapping, then the subdifferential of F is locally bounded.

This result was obtained by Valadier [13] on some special assumptions.

Graph of the GMM T is the set $GT = \{(x, A) \in X \times L(X, Y): A \in Tx\}$. T is said to be maximal if its graph is not properly contained in the graph of any other GMM. Kusraev [7] proved that the subdifferetials of convex mappings are maximal GMM.

Let us recall that the multivalued mapping $T: X \to L$ is said to be upper (resp. lower) semicontinuous at the point $x \in X$ if for every open $U \subset L$, $Tx \supset U$ (resp. $Tx \cap U \neq \emptyset$) there exists an open neighbourhood V of x such that $Tz \subset U$ (resp. $Tz \cap T \neq \emptyset$) for all $z \in V$.

The sets $U(x, y) = \{A \in L(X, Y): |\langle Ax, y^* \rangle| \leq 1\}$ form a local subbasis at 0 for a locally convex topology in $L(X, Y)$. This topology will be denoted by s and the norm topology by n.

Proposition 1. Let X be a Banach space, Y be a normed lattice and $T: X \to L(X, Y)$ be a maximal GMM with non-empty images. Then

a) T has a closed graph in $(X, n) \times (L(X, Y), s)$.

b) T has compact images in $(L(X, Y), s)$ if and only if Y has compact intervals in (Y, w).

c) If Y is reflexive, then $T: (X, n) \to (L(X, Y), s)$ is upper semicontinuous at every point of X.

Part b) of this proposition was obtained in [3] for subdifferentials of convex mappings. In the classical situations, i. e. $Y = R$, these results can be found in [6].

Proposition 2. Suppose X is a normed space, Y is a normed lattice, T is a GMM and $x \in X$. Then T is single-valued at x iff the monotone mappings $(y^* \circ T): X \to X^*$ are single-valued at x for all $y \in Y^*_+$. If $T: (X, n) \to (L(X, Y), s)$ is lower semicontinuous at the point x, then T is single-valued at that point.

If $Y = R$, this result is contained in the paper of Kenderov [4].

Proposition 3. Let X be a Banach space, Y be a separable normed lattice. If every monotone mapping $\Theta: X \to X^*$ with non-empty images is single-valued at the points of some dense G_δ subset of X, then every GMM $T: X \to L(X, Y)$ with non-empty images is single-valued at the points of a dense G_δ subset of X.

The requirement of this proposition is fulfilled if X has an equivalent strictly convex norm [6] (in particular, if X is a WCG space) or X is an Asplund space [5]. If Y is order complete, then the single-valuedness of the subdifferential ∂_F at the point x is equivalent to Gâteaux differentiability of the convex mapping F at x (see [1], p. 233). Thus Theorem 1 is a corollary of this result and Proposition 3.

Theorem 4. Let X be a normed space, Y be a normed lattice $F: X \to Y$ be a continuous convex mapping, $x \in X$ and there exists $\varepsilon > 0$ such that $\partial_F(z) \neq \emptyset$ whenever $\|z - x\| < \varepsilon$. Then F is Frechet differentiable at x if and only if the GMM $\partial_F: (X, n) \to (L(X, Y), n)$ is single-valued and upper semicontinuous at x.

Proposition 4. Let X be an order complete Banach lattice. Then Y has w-compact intervals iff every GMM with non empty images $T: R^n \to (L(R^n, Y), n)$

is single-valued and upper semicontinuous at the points of a dense G_δ subset of R^n.

We use a theorem of Kenderov [5], Proposition 1, a result of Losanovski [2], and Example 2 to prove Proposition 4. Theorem 1 is a consequence of this proposition and Theorem 4.

Example 1. Let m be the space of all bounded sequences with real terms. In the usual norm and order, m is an order complete Banach lattice. Suppose $Q=\{r_1, r_2, \dots r_n, \dots\}$ is the set of all rational numbers in $(0, 1)\subset R$. The mapping $F: R\rightarrow m$ is defined by $Fx=(|x-r_1|, |x-r_2|, \dots |x-r_n|, \dots)$. F is a continuous convex mapping. It is Gâteaux differentiable at the points of $R\setminus Q$, but F is nowhere Frechet differentiable in $(0,1)$. Let us note that $\partial_F: R \rightarrow (m, w(m, l_1))$ is upper semicontinuous at every point of R but $\partial_F: R \rightarrow (m, w(m, m^*))$ is nowhere upper semicontinuous in $(0, 1)$.

Example 2. Let l_2 be the space of all sequences $x=(x_1, x_2, \dots x_n, \dots)$ with real terms for which $\|x\|=\sum_{n=1}^{\infty} x_n^2<\infty$. With respect to this norm and usual order, l_2 is an order complete Hilbert lattice. The mapping $P: l_2\rightarrow l_2$, defined by $Px=|x|$, where $|x|=(|x_1|, |x_2|, \dots |x_n|, \dots)$ is a continuous convex mapping. F is not Frechet differentiable at any point of l_2.

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