GENERALIZED MONOTONE MAPPINGS AND DIFFERENTIABILITY OF VECTOR-VALUED CONVEX MAPPINGS

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Let $F: X \to Y$ be a convex mapping defined in the Banach space X with values in the Banach lattice Y. The question of differentiability of F almost everywhere in X (in the Baire category sense) is studied. As the examples show the problem is not trivial even in the case when X is the usual real line R. For $X = R^p$ it is shown that every convex mapping $F: X \to Y$ is Frechet differentiable at the points of some dense G_δ subset of X iff Y has weakly compact intervals. This result as well as some others are obtained as corollaries of theorems concerning generalized monotone mappings. Some examples are given which outline the theory.

0. Convex mapping with values in a vector lattice have been the subject of much research in the last years. They were studied by many authors in different directions (see [1; 7; 13; 14; 16; 19; 22; 24]). The generalized monotone mappings also got attention (see [14]).

The main purpose of this article is to prove the results announced in [13] and some new similar results, i.e. to give some conditions under which a given convex mapping $F: X \to Y$ is Frechet (Gâteaux) differentiable at the points of a dense G_δ subset of X. These differential properties of convex mappings are studied with the help of generalized monotone mappings.

Let X be a Banach space, Y be a normed lattice with a positive cone Y_+ (see [23, p. 192]) and $L = L(X, Y)$ be the space of all bounded linear mappings from X into Y. Let us recall that the mapping $F: X \to Y$ is said to be convex if $F(\alpha x_1 + (1-\alpha)x_2) \leq \alpha F x_1 + (1-\alpha) F x_2$ for all $x_1, x_2 \in X$ and $0 \leq \alpha \leq 1$. We'll suppose that the effective domain of F is the whole space X. If this is not so, all results are valid for the set int (dom F). The convex mapping F is called order bounded at the point $x \in X$ if there exist a neighbourhood V of x and $y \in Y$ such that $Fz \leq y$ for every $z \in V$. If F is order bounded at all points of X, then F is called order bounded. V a l a d i e r [22] proved that every such mapping is continuous. The convex mapping $P: X \to Y$ is said to be sublinear if it is positive homogeneous, i.e. $P(\alpha x) = \alpha P x$ for every $x \in X$ and $\alpha \in R$, $\alpha \geq 0$. The subdifferential of the convex mapping F is the following multivalued mapping:

$$\partial_F : x \to \{A \in L(X, Y) : Az - Ax \leq Fz - Fx \text{ for all } z \in X\}.$$

Sometimes subdifferentials of F are called the images of ∂_F. The support set of the sublinear mapping P is the set $\partial_P(0)$ and it is not difficult to show that $\partial_P(x) \subset \partial_P(0)$ for every $x \in X$.

D e f i n i t i o n 0.1. *We say that the multivalued mapping* $T: X \to L(X, Y)$ *is a generalized monotone mapping* (GMM) *if* $(A_1 - A_2)(x_1 - x_2) \geq 0$ *whenever* $x_i \in X$, $A_i \in T x_i$ *and* $i = 1, 2$ (*see also* K u s r a e v [14]).

In the special case when $Y=R$ the definition coincides with the well known definition of a monotone mapping or monotone operator (Browder [2], Minty [17]). The subdifferential ∂_F of a convex mapping is a GMM. The multivaled mapping $T: X \to L$ is a GMM iff the multivalued mappings $(y^* \circ T): X \to X^*$, $(y^* \circ T)x = \{y^* \circ A: A \in Tx\}$, are monotone for every $y^* \in Y_+^*$. The graph of the GMM is the set $GT = \{(x, A) \in X \times L: A \in Tx\}$. T is said to be maximal if its graph is not properly contained in the graph of any other GMM. Kusraev [14] proved that the subdifferential of a convex mapping is a maximal GMM. By means of Zorn's lemma it is not difficult to see that the graph of every GMM is contained iu the graph of some maximal GMM.

In what follows we shall consider GMM for which $Tx \neq \emptyset$ for all $x \in X$.

1. Continuity Properties of Generalized Monotone Mappings. Rockafellar [21] proved that every monotone mapping $T: X \to X^*$ is locally bounded. This result can be generalized.

Theorem 1.1. *Suppose that X is a Banach space, Y is a normed lattice and $T: X \to L(X, Y)$ is a GMM. Then T is locally bounded at every point of X, i.e. for $x_0 \in X$ there exists a neighbourhood V of x_0 such that the set $T(V) = \cup\{Tx: x \in V\}$ is a bounded subset of $L(X, Y)$.*

Proof. We use the idea of Rockafellar's proof. The following lemma is the key point for the proof of the Theorem 1.1.

Lemma. *If there exist two bounded sets: a set $S \subset X$, $\operatorname{int} \overline{\operatorname{co} S} \neq \emptyset$ and a set $U \subset L$ such that $Tx \cap U \neq \emptyset$ for every $x \in S$, Then T is locally bounded at every point of $\operatorname{int} \overline{\operatorname{co} S}$.*

Proof of the Lemma. Let $\bar{x} \in \operatorname{int} \overline{\operatorname{co} S}$. There is $\varepsilon > 0$ such that $B(\bar{x}, 2\varepsilon) \subset \overline{\operatorname{co} S}$ ($B(x, \varepsilon)$ denotes the closed ball with center x and radius ε). We fix arbitrary $x \in B(\bar{x}, \varepsilon)$, $A \in Tx$, $u \in S$ and $A_1 \in Tu \cap U$. We shall make use of the monotonicity of T. For every $y^* \in Y_+^*$, $\|y^*\| \leq 1$ we have $\langle A(u-x), y^* \rangle \leq \langle A_1(u-x) y^* \rangle \leq \|A_1\| . (\|u\| + \|x\|) . \|y^*\| \leq C$ where the constant C depends only on the sets S and U. Since the set $M = \{u \in X: \langle A(u-x), y^* \rangle \leq C\}$ is closed and convex and $S \subset M$, then $\overline{\operatorname{co} S} \subset M$. It follows that for $v \in X$ and $\|v\| \leq 1$ $x + \varepsilon/2v \in B(\bar{x}, \varepsilon) \subset M$, which is equivalent to $|\langle Av, y^* \rangle| \leq 2C/\varepsilon$. Since Y is a lattice then $|\langle Av, y^* \rangle| \leq 4C/\varepsilon$ for all $y^* \in Y^*$, $\|y^*\| \leq 1$. The last inequality shows that $\|A\| \leq 4C/\varepsilon$. The Lemma is proved.

Let us now consider the sets $S_n = \{x \in X: \|x\| \leq n \text{ and } Tx \cap B(0, n) \neq \emptyset\}$. T has nonempty images and hence $X = \cup_{n=1}^{\infty} S_n$. There exists n_0 such that $\operatorname{int} \overline{\operatorname{co} S_{n_0}} \neq \emptyset$. We apply the Lemma for $S := S_{n_0}$ and $U := B(0, n_0)$ and obtain that T is locally bounded at the points of $\operatorname{int} \overline{\operatorname{co} S_{n_0}}$. Let us now take $x \in X$, we'll prove that T is locally bounded at x. For that purpose we choose $\bar{x} \in \operatorname{int} \overline{\operatorname{co} S_{n_0}}$, $\varepsilon > 0$, $x_1 = x + \varepsilon(x - \bar{x})$ and $A_1 \in Tx_1$. Let V be such an open neighbourhood of $\bar{x}$ that $T(V)$ is a bounded subset of L. We apply again the Lemma, but now for $S := V \cup \{x_1\}$ and $U := T(V) \cup \{A_1\}$. Since $x \in \operatorname{int} \overline{\operatorname{co} S}$, T is locally bounded at the point x. The proof of the Theorem 1.1 is finished.

Corollary 1.2. *Let X be a Banach space, Y be a normed lattice, $F: X \to Y$ be a continuous convex mapping and $\partial_F(x) \neq \emptyset$ for all points x of some open set $V \subset U$. Then the subdifferential of F is locally bounded at the points of V.*

This result was obtained by Viladier [22] for order bounded convex mappings. We remind that if Y is an order complete lattice (every order bounded from below set in Y has an infinum), then every convex mapping $F: X \to Y$ has the subdifferential with nonempty image.

Further we'll need some topologies in X, Y and $L(X, Y)$. The norm topology in X, Y or L will be denoted by n. We denote by $w(Y, Z)$ the weak topology, generated by a total subset Z of Y^*, while $w=w(Y, Y^*)$. Y is called a conjugate (Banach) lattice if there exists a normed lattice E such that $Y=E^*$ and $Y_+=E_+^*$. In this case $w^*=w(Y, E)$.

When Y has a locally convex topology τ, we can define in $L=L(X, Y)$ a topology s_τ in the following way: The sets $U(x, V)=\{A \in L: Ax \in V\} x \in X$ and V belongs to the local basis at $0 \in Y$ of the topology τ, form a local subbasis at $0 \in L$. A net $\{A_\alpha\} \subset L$ converges to A in s_τ iff τ-lim $A_\alpha x = Ax$ for all $x \in X$. In addition, if $\tau=w(Y, Z)$ for some total subspace Z of Y^*, then the topology s_τ coincides with the weak topology in $L(X, Y)$, generated by $X \hat{\otimes} Z \subset L(X, Y)^*$, i.e. $s_\tau = w(L(X, Y), X \hat{\otimes} Z)$. $X \hat{\otimes} Z$ is the projective tensor product of X and Z (see [5, *p.* 227]).

The set $Z \subset Y^*$ is called ordering, if $y \in Y_+$ is equivalent to $\langle y, z\rangle \geq 0$ for all $z \in Z \cap Y_+^*$. Whenever Y is a normed lattice, Z is total. The linear subspace $Z \subset Y^*$ is said to be norming if for every $y \in Y, \|y\|>1$ there exists $z \in Z$, $\|z\| \leq 1$ such that $\langle y, z\rangle > 1$.

Proposition 1.3 *Let X be a Banach space, Y be a normed lattice, Z be an ordering subset of Y and $\tau = w(Y, X)$. Then the graph of every maximal GMM $T: X \to L(X, Y)$ is a closed subset of $(X, n) \times (L, s_\tau)$.*

Proof. Suppose that (x_α, A_α) is a convergent net in $(X, n) \times (L, s_\tau)$ and $\lim (x_\alpha, A_\alpha) = (x_0, A_0)$. This means that $\|x_\alpha - x_0\| \to 0$ and $\langle A_\alpha x, z\rangle \to \langle A_0 x, z\rangle$ for every $x \in X$ and $z \in Z$. Let $(x, A) \in GT$, $(A-A_\alpha)(x-x_\alpha) = Ax - A_\alpha x - Ax_\alpha + A_\alpha x_\alpha \geq 0$. It is obvious that $\|Ax_\alpha - Ax_0\| \to 0$ and $\tau-\lim A_\alpha x = A_0 x$. In addition, for every $z \in Z$ we have $|\langle A_\alpha x - A_0 x_0, z\rangle| \leq |\langle A_\alpha x - A_\alpha x_0, z\rangle + |\langle A_\alpha x_0 - A_0 x_0, z\rangle| \leq \|A_\alpha\| . \|x_\alpha - x_0\| . \|z\| + |\langle A_\alpha x_0 - A_0 x_0, z\rangle| \to 0$ because there exists α_0 such that the set $\{A_\alpha : \alpha \succ \alpha_0\}$ is bounded (Theorem 1.1). Consequently τ-lim $(A - A_\alpha)(x - x_\alpha) = (A - A_0)(x - x_0) \geq 0$ as the cone Y_+ is τ-closed. Due to the maximality of T, it follows that $(x_0, A_0) \in GT$. The Proposition 1.3 is proved.

Now we need the following definition. For a pair of elements y_1, y_2 in Y with $y_1 \leq y_2$, let $[y_1, y_2] = \{y \in Y: y_1 \leq y \leq y_2\}$. The sets of the form $[y_1, y_2]$ are called intervals.

Proposition 1.4. *Let X be a Banach space, Y be an order complete normed lattice, Z be an ordering and norming subspace of Y, τ be a locally convex topology in Y and $w(Y, Z) \leq \tau \leq n$. Then the following conditions are equivaleut:*

(i) *Y has τ-compact intervals.*

(ii) *Every maximal GMM $T: X \to L$ has s_τ-compact images.*

(iii) *Every continuous sublinear mapping $P: X \to Y$ has s_τ-compact support set.*

Proof. We'll prove that (i) implies (ii). Let us take a maximal GMM $T: X \to L$ and $x_0, x \in X$. The set $\{Ax: A \in Tx_0\}$ is contained in the interval $[A_1 x, A_2 x]$ for some $A_1 \in T(x_0 + x)$ and some $A_2 \in T(x_0 - x)$. Since every interval in Y is τ-compact, then $\Pi\{[A_1 x, A_2 x]: x \in X\}$ is a compact subset of $(Y, \tau)^X$ with respect to the pointwise convergence topology t. This topology coincides in $L(X, Y)\subset(Y, \tau)^X$ with the topology s_τ. Since Z is norming and $w(Y, Z)\leq\tau\leq n$, it is not difficult to see that every bounded and s_τ-closed subset of $L(X, Y)$ is t-closed in $(Y, \tau)^X$. Now Tx_0 is s_τ-closed by Proposition 1.3 and bounded by Theorem 1.1. Therefore Tx_0 is s_τ-compact.

It is obvious that (ii) implies (iii). It remains to show that (iii) implies (i). It suffices to prove that for some y_0 the interval $[-y_0, y_0]$ is τ-compact. We define $P: X\to Y$ by $Px=\|x\|y_0$. It is clear that $\{Ax_0: A\in\partial_P(0)\}\subset[-y_0, y_0]$ for some $x_0\in X$, $\|x_0\|=1$. The inclusion $\{Ax_0: A\in\partial_P(0)\}\supset[-y_0, y_0]$ is a consequence of the Hahn-Banach extension theorem for order complete vector lattice (see [1, p. 202]). The Proposition 1.4 is proved.

Every conjugate lattice Y has w^*-compact intervals and hence every maximal GMM $T: X\to L(X, Y)$ has s_{w^*}-compact images. The Banach lattice with w-compact intervals possesses a number of equivalent properties, which, are discribed in [3]. Ioffe and Levin [4] proved that the subdifferential of every continuous convex mapping $F: X\to Y$ has s_w-compact images iff Y has w-compact intervals. We get this result as a corollary of Proposition 1.4. The Banach lattices R^p, c_0, $l_p(1\leq p\leq\infty)$ have n-compact intervals.

We shall recall that the multivalued mapping $T: (X, n)\to(L, s)$ is said to be upper semicontinuous — s-u. s. c. (resp. lower semicontinuous — s-l. s. c.) at the point $x_0\in X$ if for every s-open $U\subset L$, $Tx_0\subset U$ (resp. $Tx_0\cap U\neq\emptyset$) there exists $\delta>0$ such that $Tx\subset U$ (resp. $Tx\cap U\neq\emptyset$) for all $x\in B(x_0, \delta)$. Kenderov [7, 8] proved that every maximal monotone mapping $T: X\to X^*$ is upper semicontinuous from X to X^* endowed with the weak* topology. A similar result is also valid for GMM.

Proposition 1.5. *Suppose that X is a Banach space, Y is a normed lattice, T is a maximal GMM, Z is an ordering subset of Y and $w(Y, Z)\leq\tau\leq n$. If for some $x_0\in X$ there exists a neighbourhood V of x_0 such that $T(V)$ is relativity s_τ-compact, then T is s_τ-u.s.c. at x_0.*

Proof. Let us suppose the contrary: there exist a s_τ-open set $U\subset L$, $Tx_0\subset U$, and a convergent net $\{x_\alpha\}\subset X$, $\|x_\alpha-x_0\|\to 0$, for which $A_\alpha\in Tx_\alpha/U$. When α is large enough, $x_\alpha\in V$ and there exists a convergent (in (L, s_τ)) subnet $A_{\alpha(\beta)}\to A_0$. Since the graph of T is closed (Proposition 1.3), $A_0\in Tx_0$. On the other hand, the set L/U is s_τ-closed and hence $A_0\in L\setminus U$. This contradiction proves the Proposition 1.5.

Corollary 1.6. *If X is a Banach space, Y is a conjugate lattice, then every maximal GMM $T: X\to L$ is s_{w^*}-u.s.c. at every point of X.*

Proof. Since the closed unit ball in $L(X, Y)$ is s_{w^*}-compact and T is locally bounded, we can apply Proposition 1.5.

Corollary 1.7. *Let X be a Banach space, Y be an order complete normed lattice and $P: X\to Y$ be a continuous sublinear mapping. Then if Y has w-compact (resp. n-compact) intervals, the subdifferential of P is s_w-u.s.c. (resp. s_n-u.s.c.) at every point of X.*

Proof. This is imediate from Proposition 1.5 because $\partial_P(x)\subset\partial_P(0)$ for all $x\in X$ and $\partial_P(0)$ is s_w-compact (resp. s_n-compact) by Proposition 1.4.

2. Single-valuedness of Generalized Monotone Mappings. Recall that the convex mapping $F: X\to Y$ is called Gâteaux differentiable at $x_0\in X$ if

$$F(x_0;\ h)=\inf\{\frac{F(x_0+h)-Fx_0}{\lambda}:\ \lambda>0\}$$

exists for all $h\in X$ and $F(x_0;\ .)$ is a linear mapping. When Y is an order complete lattice, F is Gâteaux differentiable at x_0 iff the subdifferential of F is single-valued at x_0 (see [1, p. 212]).

Proposition 2.1. *Suppose that X is a normed space, Y is a normed lattice, Z is a total subset of Y^*, $\tau = w(Y, Z)$, T is a GMM and $x_0 \in X$.*

(a) *T is single-valued at x_0 if and only if the monotone mappings $(z \circ T)$ are single-valued at x_0 for all $z \in Y^*_+ \cap Z$.*

(b) *If T is s_τ-l.s.c. at the point x_0, then T is single-valued at this point.*

Proof. It is clear that (a) is true. Let T be s_τ-l.s.c. at x_0 and $A_1, A_2 \in Tx_0$, $A_1 \neq A_2$. Choose $x \in X$, $\|x\| < 1$ and $z \in Z \cap Y^*_+$ such that $\langle A_1 x, z\rangle < \langle A_2 x, z\rangle$. The set $U = \{A \in L: \langle Ax, z\rangle < \langle A_2 x, z\rangle\}$ is s_τ-open and $A_1 \in Tx_0 \cap U \neq \emptyset$. There exists $\delta > 0$ such that $Tx' \cap U \neq \emptyset$ whenever $\|x' - x_0\| < \delta$. Let $x' = x_0 + \delta/2 . x$ and by the monotonicity of $TA'x \geq A_2 x$ for all $A' \in Tx'$. But now $z \in Y^*_+$ and $\langle A'x, z\rangle \geq \langle A_2 x, z\rangle$, hence $A' \notin U$. This contradiction completes the proof.

In the case $Y = R$, this result is contained in the paper of Kenderov [12].

Theorem 2.2. *Let X be a Banach space, Y be such a normed lattice that there exists $Z \subset Y^*$ for which $Z \cap Y^*_+$ is weak*-separable and total. If every monotone mapping $\theta: X \to X^*$ is single-valued at the points of some dense G_δ subset of X, then every GMM $T: X \to L(X, Y)$ is single-valued at the points of a dense G_δ set in X.*

Proof. Let Z_0 be a countable and w^*-dense subset of $Z \cap Y^*_+$. For every $z \in Z_0$ the monotone mapping $(z \circ T): X \to X^*$ is single-valued on the set $G(z)$ which is a dense G_δ subset of X. It follows that the set $G = \cup\{G(z): z \in Z_0\}$ is also dense G_δ. Since Z_0 is total the GMM T is single-valued on G. Theorem 2.2 is proved.

A Banach space X is called an Asplund space) (weak Asplund space) if every continuous convex real-valued function on an open convex subset of X is Frechet (Gâteaux) differentiable in a dense G_δ subset of its domain (see [5, 15, 18, 20]). The requirements of Theorem 2.2 are fulfilled if X is an Asplund space (see Theorem 3.2). Kenderov [11] proved that if X has an equivalent strictly convex norm (i. e. if $x_1, x_2 \in X$, $\|x_1\| = \|x_2\|$ and $x_1 \neq x_2$ then $\|(x_1 + x_2)/2\| < \|x_1\|$) then every monotone mapping $\theta: X \to X^*$ is single-valued on a dense G_δ set. In particular, Theorem 2.2 ~~holds for~~ every separable space X because every such space has an equivalent dual strictly convex norm. The above mentioned theorem is valid if Y is separable or Y is a conjugate lattice, $Y = E^*$ and E is separable.

Theorem 2.3. *Let X be a weak Asplung space, Y be such an order complete normed lattice that there exists $Z \subset Y^*$, for which $Z \cap Y^*_+$ is weak separable and total. Then every continuous convex mapping $F: X \to Y$ is Gâteaus differentiable at the points of a dense G_δ set in X.*

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Proof. Take the proof of Theorem 2.2 and substitute ∂_F for T. We note that there is a continuous convex mapping $F: R \to Y$ which is not Gâteaux differentiable at any point of some open interval in R (Example 4.1) and a continuous convex function $f: X \to R$ which is nowhere Gâteaux differentiable on X (Phelps [20]).

3. Single-valuedness and Norm-to-norm Upper Semicontinuity of Generalized Monotone Mappings. The single-valued mapping $F: X \to Y$ is called Frechet differentiable at the point $x_0 \in X$ with derivative $A \in L(X, Y)$ if

$$\lim_{\|h\|\to 0}\frac{\|F(x_0+h)-Fx_0-Ah\|}{\|h\|}=0.$$

Theorem 3.1. *Let X be a normed space, Y be a normed lattice, $F: X\to Y$ be a continuous convex mapping, $x_0\in X$ and there is $\varepsilon_0>0$ such that $\partial_F(x)\neq\emptyset$ whenever $\|x-x_0\|<\varepsilon_0$. Then F is Frechet differentiable at x_0 if and only if the GMM ∂_F is single-valued and n-u.s.c. at x_0.*

Proof. Suppose that $A_0\in L$ is the Frechet derivative of F at x_0 and $0<\varepsilon<\varepsilon_0$. We'll find $\delta>0$ such that if $h\in X$, $\|h\|<\delta$, then $\|A-A_0\|<\varepsilon$ for every $A\in\partial_F(x_0+h)$. There exists such $\eta>0$, that $\|y_1\vee y_2\|<\varepsilon/3(y_1, y_2\in Y$ and $y_1\vee y_2=\sup\{y_1, y_2\})$ whenever $\|y_1\|$, $\|y_2\|<\eta$. Now there exists $\zeta>0$ such that for any $z\in X$, $\|z\|<\zeta$ we have

$$\frac{\|F(x_0+z)-Fx_0-A_0z\|}{\|z\|}<\frac{\eta}{3}.$$

We choose $\delta=\zeta/8$ and fix $h\in X$, $\|h\|<\delta$ and $A\in\partial_F(x_0+h)$. It suffices to show hat $\|A-A_0\|<\varepsilon$. For that purpose we take $g\in X$, $\|g\|=2\delta$. The following nequalities hold:

$$F(x_0+h)-Fx_0\geq A_0h,$$

$$F(x_0+h+g)-F(x_0+h)\geq Ag$$

Hence $Ag\leq F(x_0+h+g)-F(x_0+h)+Fx_0-Fx_0\leq F(x_0+h+g)-Fx_0-A_0h$, so $(A-A_0)g\leq F(x_0+h+g)-Fx_0-A_0(h+g)$. Let us denote the right-hand side of the last inequality by $\Phi(g)$. If we replace g by $-g$, we obtain $\Phi(-g)\geq(A-A_0)(-g)$. Thus

$$-\Phi(-g)\leq(A-A_0)g\leq\Phi(g).$$

Since $\Phi(g)\geq 0$ and $\Phi(-g)\geq 0$, $|(A-A_0)g|\leq\Phi(g)\vee\Phi(-g)$. Hence $\|(A-A_0)g\|\leq\|\Phi(g)\vee\Phi(-g)\|$. Now we have

$$\frac{\|\Phi(-g)\|}{\|g-h\|}<\frac{\eta}{3},\quad \frac{\|\Phi(g)\|}{\|g-h\|}=\frac{\|\Phi(g)\|}{\|g+h\|}\cdot\frac{\|g+h\|}{\|g-h\|}<\frac{\eta}{3}\cdot\frac{\|g\|+\|h\|}{\|g\|-\|h\|}<\eta.$$

It follows that

$$\frac{\|(A-A_0)g\|}{\|g-h\|}\leq\frac{\|\Phi(g)\|}{\|g-h\|}\vee\frac{\|\Phi(-g)\|}{\|g-h\|}<\frac{\xi}{3},\ \text{hence}\ \frac{\|(A-A_0)g\|}{\|g\|}=\frac{\|(A-A_0)\|}{\|g-h\|}\cdot\frac{\|g-h\|}{\|g\|}<\varepsilon.$$

This shows that $\|A-A_0\|<\varepsilon$.

Let now the subdifferential of F be single-valued and n-u.s.c. at the point x_0, i. e. $\partial_F(x_0)=\{A_0\}$ and for any $\varepsilon<0$ there exists $\delta>0$ such that $\|A-A_0\|<\varepsilon$ whenever $\|h\|<\delta$ and $A\in\partial_F(x_0+h)$. For every $g\in X$ we have $F(x_0+h+g)-F(x_0+h)\geq Ag$ and hence for $g=-hF(x_0)-F(x_0+h)\geq -Ah$. Thus $0\leq F(x_0+h)-Fx_0-A_0h\leq(A-A_0)h$ and $\|F(x_0+h)-Fx_0-A_0h\|\leq\|(A-A_0)h\|\leq\|A-A_0\|.\|h\|$. The last inequality shows that F is Frechet differentiable at x_0. The proof is finished.

As it was already mentioned, the Banach space X is called an Asplund space if every continuous convex function $f: X\to R$ is Frechet differentiable at the points of some dence G_δ subset of its domain. Further we need the following well known theorem.

Theorem 3.2. *Any one of the following statements about Banach space X implies all the others:*

(a) *X is an Asplund space.*

(b) *Every monotone mapping $T: X \to X^*$ is single-valued and norm-to-norm upper semicontinuous at the points of a dense G_δ subset of X.*

(c) *Every separable subspace of X has a separable dual.*

(d) *X^* has the Radon-Nikodym property (for the definition see [5]).*

The proof of the part of this theorem (a)$\Leftrightarrow$(c) may be found in [20]. Kenderov [10; 9] proved that (a)$\Rightarrow$(b) and (c)$\Rightarrow$(b). It is trivial that (b)$\Rightarrow$(a). The equivalence between (d) and (c) may be found in [5].

Further we'll give some conditions under which a given GMM $T: X \to L(X, Y)$ is single-valued and n-u.s.c. on a dense G_δ subset of X. In the general case this problem is very complicated. There is a convex mapping $F: R \to Y$, which is not Frechet differentiable at any point of an open interval (Example 4.2 (a)), thus the GMM $\partial_F: R \to L(R, Y)$ is not n-u.s.c. at any point of this interval (Theorem 3.1). Similarly there exists a sublinear mapping P, defined in a Hilbert lattice Y with values in Y, which is Frechet differentiable at no point of Y (Example 4.3). First we shall solve the problem supposing X is a finite dimensional space, and second we'll find some requirements concerning the spaces Y and $L(X, Y)$ and (or) the GMM T under which T is single-valued and n-u.s.c. on a dense G_δ set in X. To do this we use the following corollary of a Kenderov's theorem [9, Theorem 2.1].

Theorem 3.3 (Kenderov). *Let X be a Banach space, V be a normed space and W be a norming subspace of V^*. Suppose $T: X \to (V, w(V, W))$ is an upper semicontinuous (multivalued) mapping with $w(V, W)$-compact and convex images. If*

(a) $W = V$ or

(b) *V is a conjugate space, i. e. $V = E^*$, and $W = E$, E is an Asplund space*

then there exists a dense G_δ subset G in X at every point x of which the following "continuity property" (c. p.) is fulfilled:

$$\text{(c.p)} \quad \begin{cases} \text{for every } \varepsilon > 0 \text{ there is } \delta > 0 \text{ such that } \inf\{\|v' - v''\| : \\ v' \in Tx',\ v'' \in Tx''\} \leq \varepsilon \text{ whenever } \|x' - x\| < \delta \text{ and } \|x'' - x\| < \delta. \end{cases}$$

Proposition 3.4. *Let X be a normed space, Y be a normed lattice and T be a GMM. Then T has (c.p.) at the point $x_0 \in X$ iff T is single valued and n-u.s.c. at x_0.*

Proof. If T is single-valued and n-u.s.c. at x_0, then it is not difficult to see that T has (c.p.) at x_0. Let now T has (c.p.) at the point $x_0 \in X$ and $\varepsilon > 0$. There exists $\delta > 0$ such that

$$\inf\{\|A' - A''\| : A' \in Tx',\ A'' \in Tx''\} < \varepsilon/4$$

whenever $x', x'' \in B(x_0, \delta)$. We'll prove that diam $T(B(x_0, \delta)) < \varepsilon$ and that's enough for single-valuedness and n-upper semicontinuity of T at x_0.

Let $x_1, x_2 \in B(x_0, \delta)$, $A_1 \in Tx_1$, $A_2 \in Tx_2$, $e \in X$ and $\|e\| = 1$. For some $t > 0$ we have $x' = x_1 + te \in B(x_0, \delta)$ and $x'' = x_2 - te \in B(x_0, \delta)$ and let $A' \in Tx'$, $A'' \in Tx''$, $\|A' - A''\| < \varepsilon/2$. Using the monotonicity of T we get $(A' - A_1)e \leqq 0$ and $(A_2 - A'')e \geqq 0$. Therefore $(A_1 - A_2)e \leqq (A' - A'')e$. If we take $-e$ in the place of e we obtain $(A_1 - A_2)e \geqq (A_1' - A_2'')e$ where $A_1' \in T(x_1 - te)$ and $A_2'' \in T(x_2 + te)$ are such that $\|A_1' - A_2''\| < \varepsilon/2$. Hence for every $y^* \in Y^*_+$, $\|y^*\| < 1$ we have $|\langle (A_1 - A_2)e, y^* \rangle| < \varepsilon/2$. Therefore $|\langle (A_1 - A_2)e, y^* \rangle| < \varepsilon$ for all $y^* \in Y^*$, $\|y^*\| < 1$. It follows that $\|(A_1 - A_2)e\| < \varepsilon$ and Proposition 3.4 is proved.

For usual monotone mappings this was proved in [9].

Besides the theorem of Kenderov we make use of a result of Losanowski (see [3]), Proposition 3.4 and Example 4.2 to prove the following:

Theorem 3.5. *If Y is an order complete Banach lattice, then the following assertions are equivalent.*

(i) *Y has w-compat intervals.*

(ii) *Every GMM $T: R \to L(R, Y)$ is single-valued and n.-u.s.c. at the points of some dense G_δ subset of R.*

(iii) *Every convex mapping $F: R \to Y$ is Frechet differentiable at the points of a dense G_δ subset of R.*

Proof. We note that the space $L(R, Y)$ is isometric to Y. (i) implies (ii). Let $T: R \to Y$ be a GMM and $\widehat{T}: R \to Y$ be a maximal GMM such that $Tr \subset \widehat{T}r$ for every $r \in R$. If we take $r_i \in R$, $y_i \in \widehat{T}r_i$, $i=1, 2$, the monotonicity of $\widehat{T}$ is expressed in $y_1 \geqq y_2$ whenever $r_1 \geqq r_2$. Let $r_0 \in R$, $\delta > 0$, $V=(r_0-\delta, r_0+\delta)$, $y_1 \in \widehat{T}(r_0-\delta)$, $y_2 \in \widehat{T}(r_0+\delta)$, then $\widehat{T}(V) \subset [y_1, y_2]$. Proposition 1.5 shows that $\widehat{T}$ is w-u.s.c. at r_0. $\widehat{T}$ has w-compact (Proposition 1.4) and convex images then by Theorem 3.3 (a) and Proposition 3.4 T is single-valued and n-u.s.c. at the points of some dense G_δ subset of R. T has the same property because $Tr = \widehat{T}r$ whenever $\widehat{T}r$ contains one point.

(ii) implies (iii). The convex mapping $F: R \to Y$ is continuous at every point of R because it is order bounded (see Valadier [22]). Since Y is an order complete lattice, $\partial_F(x) \neq \emptyset$ for every $x \in X$, and we can apply Theorem 3.1.

(iii) implies (i). Losanowski proved that Y has w-compact intervals iff the space m (of all bounded sequences) does not embed into Y (see [3]). There is a convex mapping $F: R \to m$ which is not Frechet differentiable at any point of the interval $(0,1) \subset R$ (see Example (4.2). Thus the implication and Theorem 3.5 are proved.

We mention that all sublinear mappings $P: R \to Y$ are Frechet differentiable at every point of $R/\{0\}$, provided Y is a normed lattice. This is not true even if R is replaced by R^2 (see Example (4.2).

Theorem 3.6. *If Y is an order complete Banach lattice and $p \geqq 2$, then the following assertions are equivalent:*

(i) *Y has w-compact intervals.*

(ii) *Every GMM $T: R^p \to L(R^p, Y)$ is single-valued and n-u.s.c. at the points of a dense G_δ subset of R^p.*

(iii) *Every convex mapping $F: R^p \to Y$ is Frechet differentiable at the points of some dense G_δ subset of R^p.*

(iv) *Every sublinear mapping $P: R^p \to Y$ is Frechet differentiable at the points of a dense G_δ subset of R^p.*

Proof. It is not difficult to show that $T(V)$ is order bounded for every bounded set $V \subset R^p$. As well as in the proof of Theorem 3.5 we get that (i) implies (ii). It is obvious that (ii) implies (iii) and (iii) implies (iv). Example 4.2 (b) enables us to get (iv) implies (i).

Thus, if $X=R^p$ we give the necessary and sufficient condition for a GMM $T: X \to L(X, Y)$ to be single-valued and n-u.s.c. at the points of a dense G_δ set in X. An analogous theorem can be obtained from the known results for Asplund spaces (see Theorem 3.2).

Theorem 3.7. *Each of the following assertions abaut a Banach space X implies all the others.*

(i) *X is an Asplund space.*

(ii) *Every GMM $T: X \to L(X, R^p)$ is single-valued and n-u.s.c. at the points of some dense G_δ subset of X.*

(iii) *Every continuous convex mapping $F: X \to R^p$ is Frechet differentiable at the points of a dense G_δ set in X.*

Proof. Let $Fx=(f_1(x), f_2(x), \ldots, f_p(x))$ and $f_i: X \to R, i=1, 2 \ldots$ We note that F is a convex mapping iff all f_i are convex functions. Moreover F is Frechet differentiable at $x \in X$ iff f_i is Frechet differentiable at x for all i. Since $L(X, R^p)=(X^*)^p$, $T=(T_1, T_2, \ldots, T_p)$, $T_i: X \to X^*$ and T_i is the monotone mapping for every $i=1, 2, \ldots p$. And the GMM $T: X \to L(X, R^p)$ is single-valued and n-u.s.c. at x if and only if the monotone mappings T_i, $i=1, 2, \ldots, p$ are single-valued and n-u.s.c. at x. In such a way this theorem is a corollary of Theorem 3.2.

Now we give some results for infinite dimensional case.

Theorem 3.8. *Let X be a Banach space, Y be a conjugate lattice and $L(X, Y)=K(X, Y)^{**}$($K(X, Y)$ is the space of all compact linear mappings from X into Y). Then every GMM $T: X \to K(X, Y)$ is single-valued and n-u.s.c. at the points of a dense G_δ subset of X.*

Proof. Suppose $Y=E^*$. Since $L(X, Y)=(X\hat{\otimes}E)^*=K(X, Y)^{**}$ [5, p. 230], the GMM $T: X \to K(X, Y)$ is $w(K(X, Y), X\hat{\otimes}E)$-u.s.c. at all points of X by Corollary 1.6. According to the Theorem 3.3 and Proposition 3.4 we get that the GMM T is single-valued and n-u.s.c. on a dense G_δ subset of X

Corollary 3.9. *If the conditions of Theorem 3.8 hold, then every continuous convex mapping $F: X \to Y$ with the property $\partial_F(x) \subset K(X, Y)$ for all $x \in X$ is Frechet differentiable at the points of a dense G_δ subset of X.*

Proof. Apply Theorem 3.8. ∂_F has nonempty images because every conjugate lattice is order complete (see [23, p. 278]).

Corollary 3.10. *Suppose X is a Banach space, Y is a conjugate lattice with n-compact intervals and $L(X, Y)=K(X, Y)^{**}$. Then every order bounded convex mapping $F: X \to Y$ is Frechet differentiable at the points of a dense G_δ subset of X.*

Proof. Since Y has n-compact intervals and F is order bounded, then $\partial_F(x) \subset K(X, Y)$ and Corollary 3.9 can be applied.

We note that if X and Y are reflexive and X or Y has the approximation property (for the definition see [5, p. 238]), then $L(X, Y)=K(X, Y)^{**}$ (Feder, Saphar [6]). In particular, Corollary 3.10 is valid for $X=l_p$, $Y=l_q$, $1<p<\infty$, $1<q<\infty$.

Theorem 3.11. *Let X be a Banach space and Y be a conjugate lattice such that $L(X, Y)$ has the Radon-Nikodym property. Then every GMM $T: X \to L(X, Y)$ is single-valued and n-u.s.c. at the points of a dense G_δ subset of X.*

Proof. Since $Y \doteq E^*$, $L(X, Y)=(X\hat{\otimes}E)^*$ and $L(X, Y)$ has the Radon-Nikodym property, then $X\hat{\otimes}E$ is an Asplund space. Theorem 3.2 (b) and Proposition 3.4 show that T is single-valued and n-u.s.c. on a dense G_δ set in X.

Corollary 3.12. *Suppose X is a Banach space, Y is a conjugate lattice such that $L(X, Y)$ has the Radon-Nikodym property. Then every continuous convex mapping $F: X \to Y$ is Frechet differentiable at the points of a dense G_δ subset of X.*

Proof. This follows immediate from Theorem 3.11.

Diestel and Morrison [4] proved that if X and Y are separable Banach spaces with the Radon-Nikodym property and $L(X, Y)=K(X, Y)$, then $L(X, Y)$ has the Radon-Nikodym property. In particular, the requirement of Theorem 3.11 and Corollary 3.12 are fulfilled if $X=l_p$, $Y=L_q$, $2\leq q<p<\infty$ or $X=L_p$, $Y=l_q$, $1\leq q<2\leq p<\infty$ (see [8]).

4. Examples. Example 4.1. (a) Let $Y=m\,[0, 1]$ be the space of all bounded functions, defined in the interval $[0,1]\subset R$. Y is an order complete Banach lattice with respect to the norm $\|y\|=\sup\{|y(t)|: t\in[0, 1]\}$ and the positive cone $Y_+=\{y\in Y: y(t)\geq 0$ for every $t\in[0, 1]\}$. We define $F: R\to Y$ in the following way: $(Fr)(t)=|r-t|$ for every $r\in R$ and $t\in[0, 1]$. F is a convex mapping. It is not difficult to see that for every $r_0\in(0,1)$ the subdifferential of F has the form: $\partial_F(r_0)-\{y_\alpha: \alpha\in[-1, 1]\}$, where

$$y(t)=\begin{cases} 1 & \text{for } t\in[0, r_0) \\ \alpha & \text{for } t=r_0 \\ -1 & \text{for } t\in(r_0, 1]. \end{cases}$$

It follows that F is nowhere Gâteaux differentiable at any point of $(0,1)$.

(b) Let now $Y=L_\infty[0, 1]$. Y is again an order complete Banach lattice with respect to the usual norm and order. The same mapping F is convex and the form of the subdifferentials, given in (a), shows that F is Gâteaux differentiable at all points of R. We'll show that F is not Frechet differentiable at any point of the interval $(0,1)$. Indeed, when $r_0\in(0,1)$, $\delta>0$, $r_0+\delta/2<1$, $h=\delta/2$ and $\partial_F(r_0)=\{y\}$, we have $\|F(r_0+h)-Fr_0-hy\|=\operatorname{ess\,sup}\{\||r_0+h-t|-|r_0-t|-hy(t)|: t\in[0, 1]\}\geq\operatorname{ess\,sup}\{||r_0+h-t|-|r_0-t|-hy(t)|: t\in(r_0, r_0+\delta/4)\}=\operatorname{ess\,sup}\{2(r_0+h-t): t\in(r_0, r_0+\delta/4)\}\geq h$.

(c) Let us note that we can consider F as a mapping from R into the space of all continuous functions $C[0,1]$. Relative to the usual norm and order $C[0,1]$ is a Banach lattice, but it is not order complete. In this situation F has empty subdifferentials at every point of $(0,1)$ (see also Ioffe and Levin [7]).

Example 4.2. (a) Let m be a space of all bounded sequences with real terms. In the usual norm and order, m is an order complete Banach lattice. Suppose that $Q=\{q_1, q_2, \ldots, q_k, \ldots\}$ is the set of all rational numbers in $(0, 1)\in R$. The mapping $F: R\to m$ is defined by

$$Fr=(|r-q_1|, |r-q_2|, \ldots, |r-q_k|, \ldots).$$

As in Example 4.1 we can prove that F is not Gâteaux differentiable at the points of Q and F is not Frechet differentiable at any point of $(0,1)$.

(b) Let us define $P: R^2\to m$ in a such way:

$$P(r, s)=(|r-sq_1|, |r-sq_2|, \ldots, |r-sq_k|, \ldots).$$

P is a sublinear mapping and it is not Frechet differentiable at any point of the open nonempty set $G=\{(r, s)\in R^2: 1/2<r/s<1$ and $1<s<2\}$.

Example 4.3. Let $l_p(1\leq p<\infty)$ (resp. c_0) be the space of all sequences $x=(x_1, x_2, \ldots x_k, \ldots)$ of real numbers for which $\|x\|_p=(\Sigma_{k=1}^\infty|x_k|^p)^{1/p}<\infty$ (resp. $x_k\to 0$ and $\|x\|=\sup_k|x_k|$). With respect to this norm and the usual order l_p and c_0 are order complete Banach lattices. We'll denote by X either the space c_0 or some of the spaces l_p. The mapping $P: X\to X$, defined by $Px=|x|$, where $|x|=(|x_1|, |x_2|, \ldots, |x_k|, \ldots)$. is a continuous sublinear mapping. It is not difficult to show that the set $A \in L(X, X)$: $A=\{a_{ij}\}$, $a_{ii} \in [-1, 1]$, $a_{ij}=0$ whenever $i \neq j$ is the support set of P. Here $\{a_{ij}\}$ is an infinite matrix, defined by the linear mapping $A \in L(X, X)$. P is Gâteaux differentiable at the points of the set $G=\{x \in X: x_k \neq 0$ for every $k\}$ which is a dense G_δ subset of X and if $x \in G$ then $\partial_P(x)=\{A\}$, $\partial_P(0) \in A=\{a_{ij}\}$, $a_{ii}=\operatorname{sgn}(x_i)$, $i=1, 2, \ldots$. P is nowhere Frechet differentiable because if $h_k=(0, 0, \ldots -2x_k, 0, \ldots) \in X$, then $\|h_k\| \to 0$ but

$$\frac{\|P(x+h_k)-Px-Ah_k\|}{\|h_k\|}=1.$$

It is interesting to note that ∂_P is s_n-u.s.c. at all points of X (Corollary 1.7 but ∂_P is not n-u.s.c. at any point of X (Theorem 3.1).

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