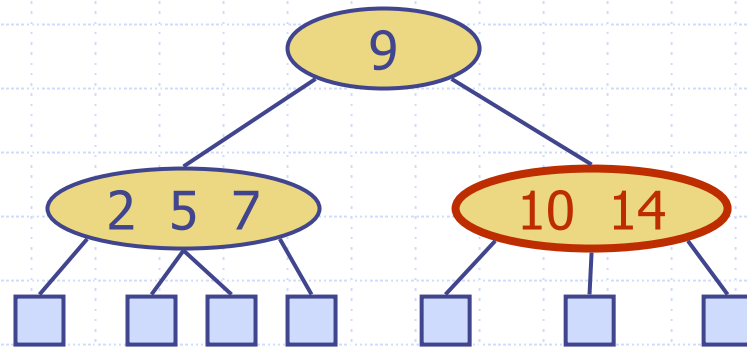


(2,4) Trees



Outline and Reading

◆ Multi-way search tree (§9.3)

- Definition
- Search

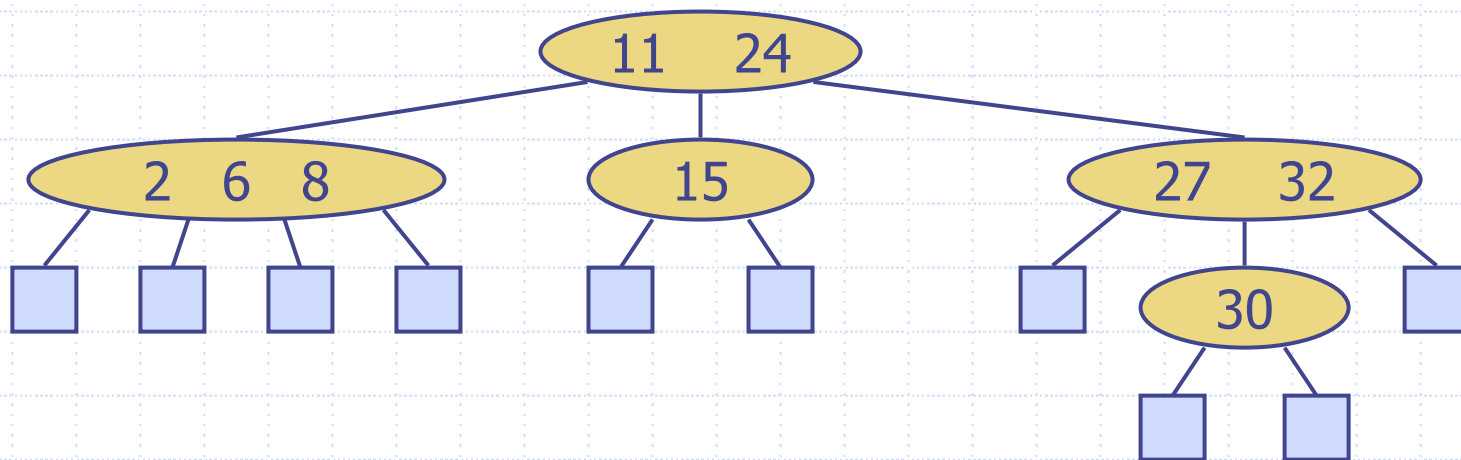
◆ (2,4) tree (§9.4)

- Definition
- Search
- Insertion
- Deletion

◆ Comparison of dictionary implementations

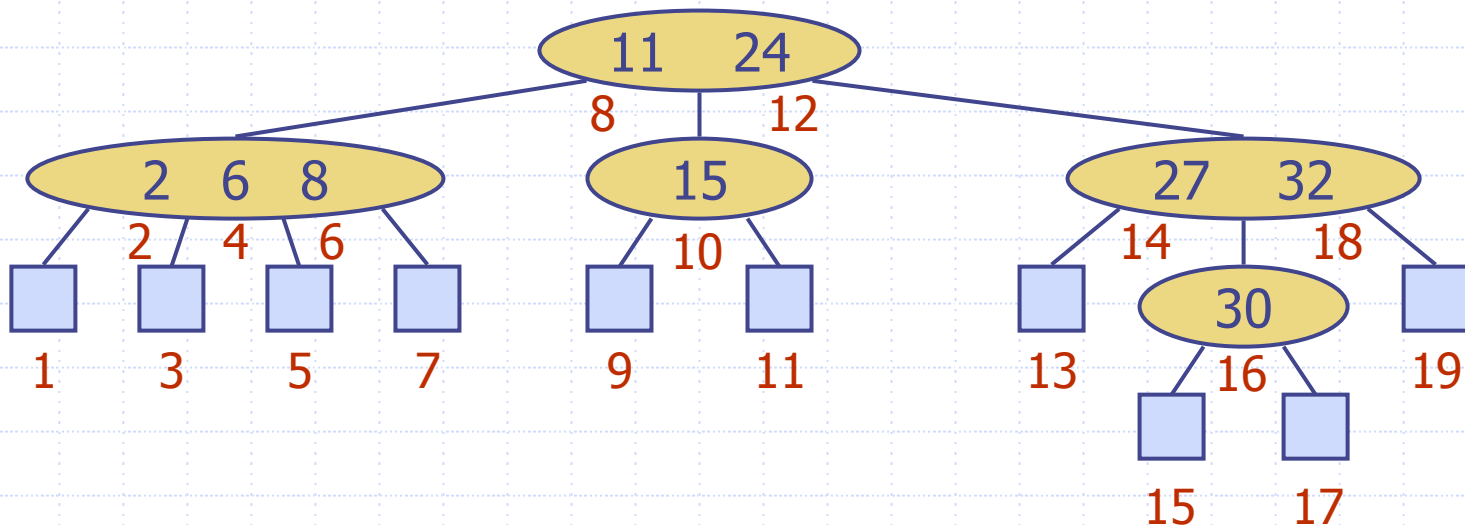
Multi-Way Search Tree

- ◆ A multi-way search tree is an ordered tree such that
 - Each internal node has at least two children and stores $d-1$ key-element items (k_i, o_i) , where d is the number of children
 - For a node with children $v_1 v_2 \dots v_d$ storing keys $k_1 k_2 \dots k_{d-1}$
 - ◆ keys in the subtree of v_1 are less than k_1
 - ◆ keys in the subtree of v_i are between k_{i-1} and k_i ($i = 2, \dots, d-1$)
 - ◆ keys in the subtree of v_d are greater than k_{d-1}
 - The leaves store no items and serve as placeholders



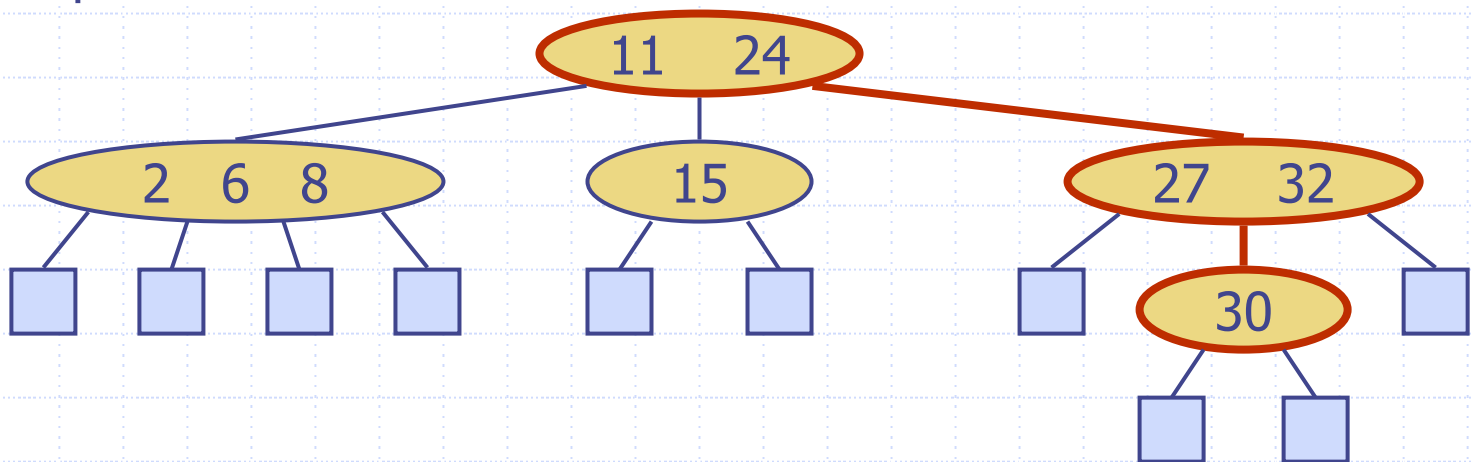
Multi-Way Inorder Traversal

- ◆ We can extend the notion of inorder traversal from binary trees to multi-way search trees
- ◆ Namely, we visit item (k_i, o_i) of node v between the recursive traversals of the subtrees of v rooted at children v_i and v_{i+1}
- ◆ An inorder traversal of a multi-way search tree visits the keys in increasing order



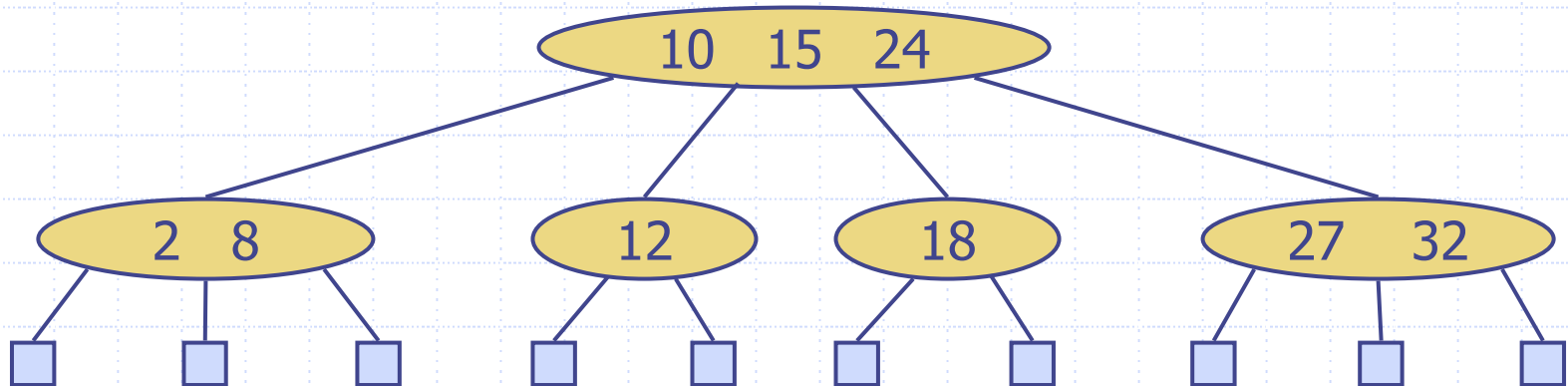
Multi-Way Searching

- ◆ Similar to search in a binary search tree
- ◆ At each internal node with children $v_1 v_2 \dots v_d$ and keys $k_1 k_2 \dots k_{d-1}$
 - $k = k_i$ ($i = 1, \dots, d - 1$): the search terminates successfully
 - $k < k_1$: we continue the search in child v_1
 - $k_{i-1} < k < k_i$ ($i = 2, \dots, d - 1$): we continue the search in child v_i
 - $k > k_{d-1}$: we continue the search in child v_d
- ◆ Reaching an external node terminates the search unsuccessfully
- ◆ Example: search for 30



(2,4) Tree

- ◆ A (2,4) tree (also called 2-4 tree or 2-3-4 tree) is a multi-way search with the following properties
 - **Node-Size Property:** every internal node has at most four children
 - **Depth Property:** all the external nodes have the same depth
- ◆ Depending on the number of children, an internal node of a (2,4) tree is called a 2-node, 3-node or 4-node



Height of a (2,4) Tree

◆ **Theorem:** A (2,4) tree storing n items has height $O(\log n)$

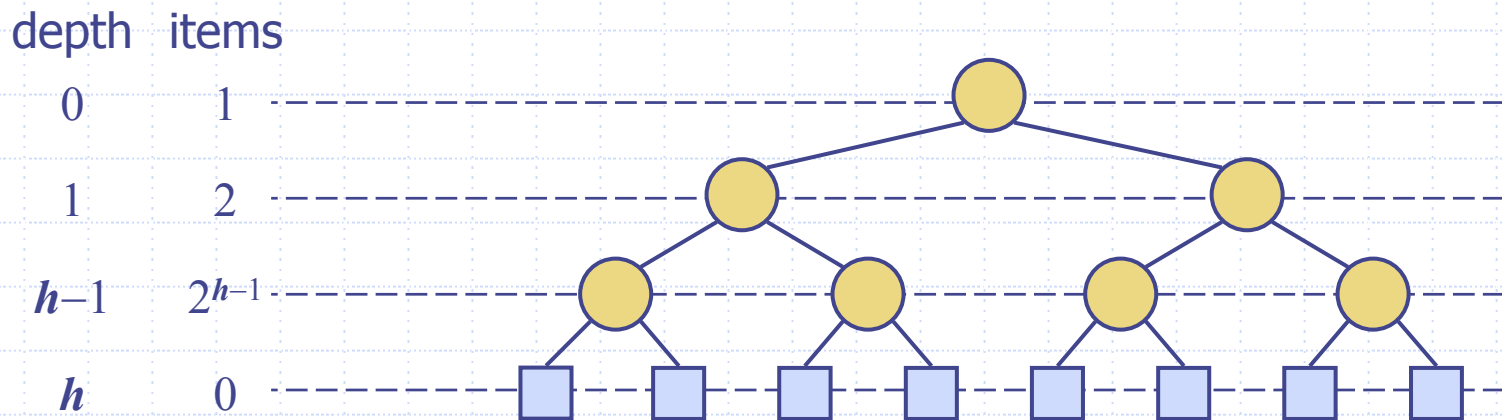
Proof:

- Let h be the height of a (2,4) tree with n items
- Since there are at least 2^i items at depth $i = 0, \dots, h-1$ and no items at depth h , we have

$$n \geq 1 + 2 + 4 + \dots + 2^{h-1} = 2^h - 1$$

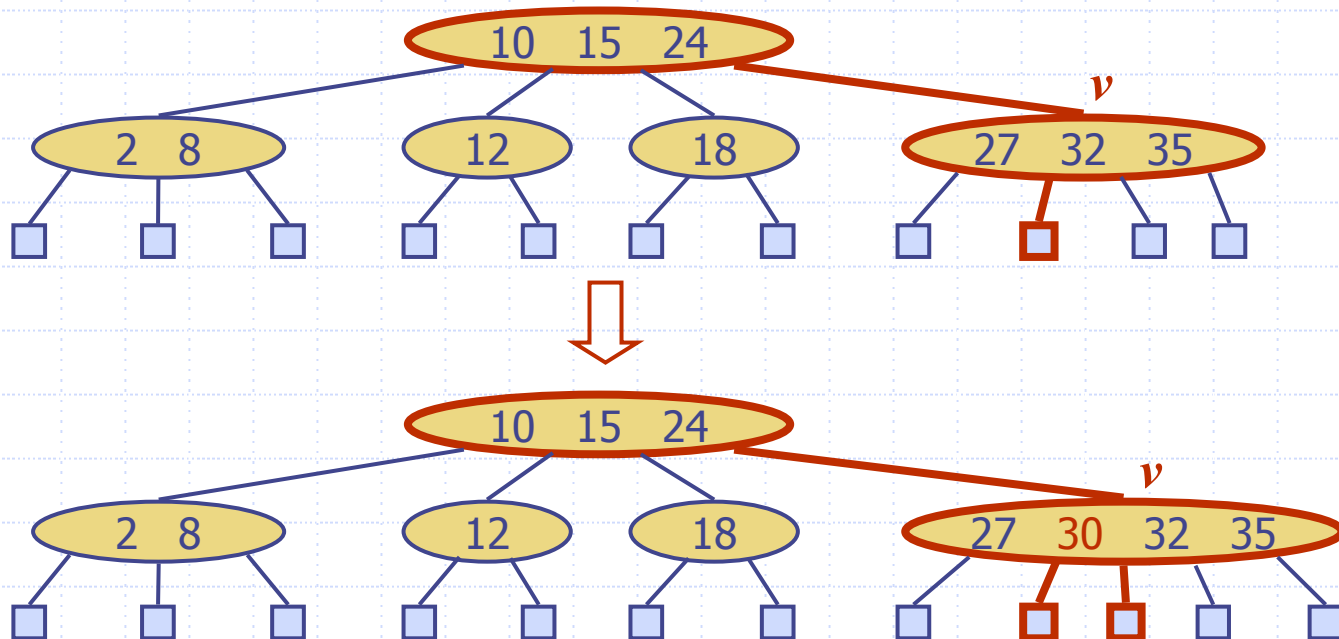
- Thus, $h \leq \log(n + 1)$

◆ Searching in a (2,4) tree with n items takes $O(\log n)$ time



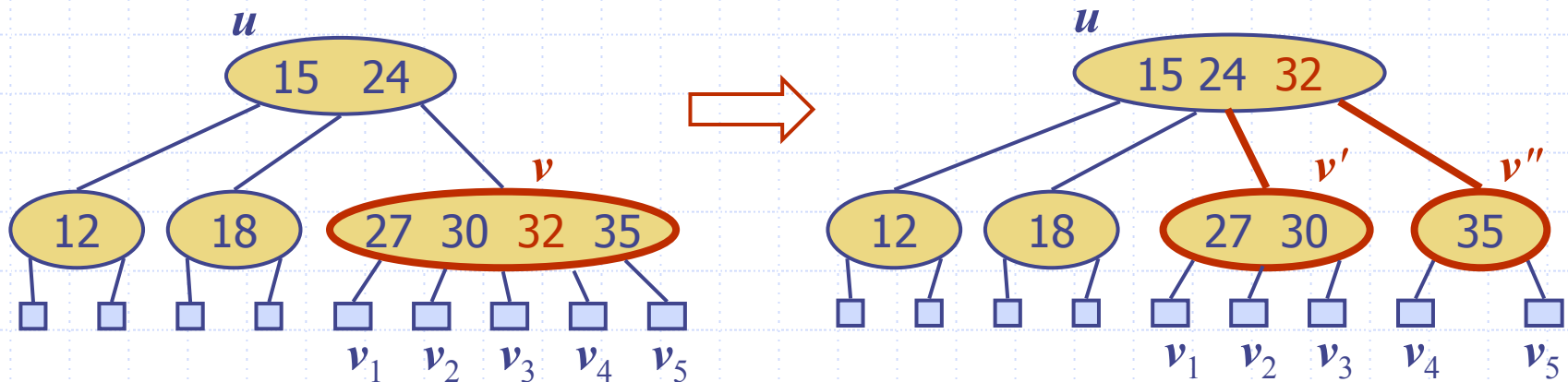
Insertion

- ◆ We insert a new item (k, o) at the parent v of the leaf reached by searching for k
 - We preserve the depth property but
 - We may cause an **overflow** (i.e., node v may become a 5-node)
- ◆ Example: inserting key 30 causes an overflow



Overflow and Split

- ◆ We handle an **overflow** at a 5-node v with a **split operation**:
 - let $v_1 \dots v_5$ be the children of v and $k_1 \dots k_4$ be the keys of v
 - node v is replaced nodes v' and v''
 - ◆ v' is a 3-node with keys $k_1 k_2$ and children $v_1 v_2 v_3$
 - ◆ v'' is a 2-node with key k_4 and children $v_4 v_5$
 - key k_3 is inserted into the parent u of v (a new root may be created)
- ◆ The overflow may propagate to the parent node u



Analysis of Insertion

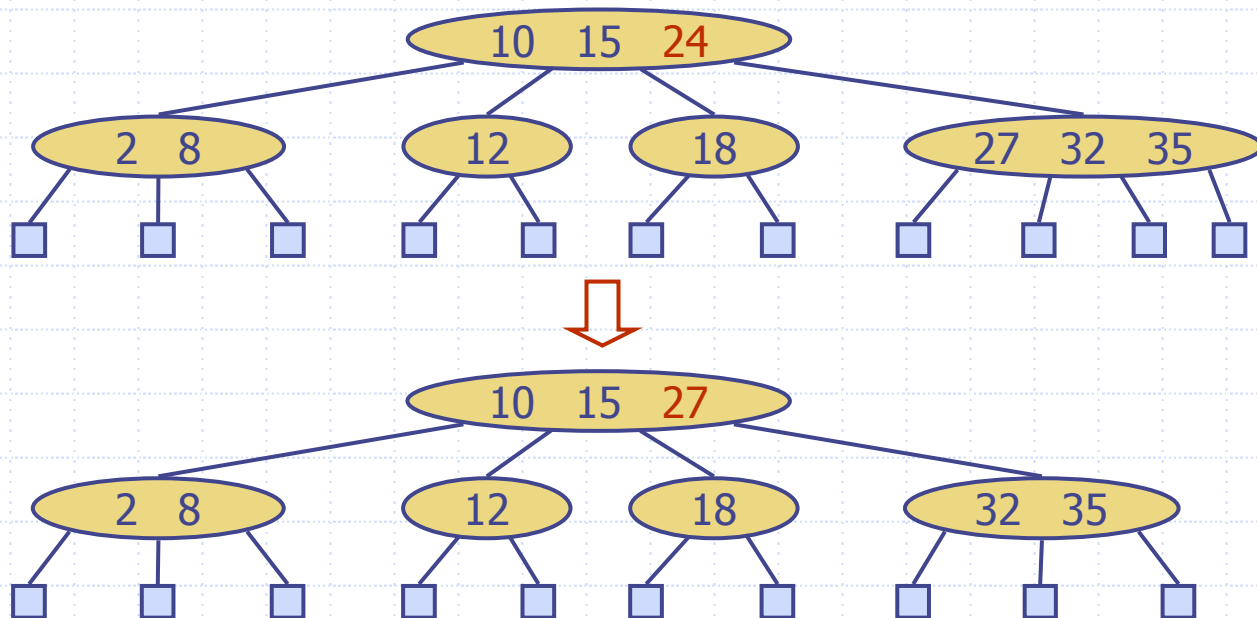
Algorithm *insertItem(k, o)*

1. We search for key k to locate the insertion node v
2. We add the new item (k, o) at node v
3. **while** *overflow*(v)
 if *isRoot*(v)
 create a new empty root above v
 $v \leftarrow \textit{split}(v)$

- ◆ Let T be a (2,4) tree with n items
 - Tree T has $O(\log n)$ height
 - Step 1 takes $O(\log n)$ time because we visit $O(\log n)$ nodes
 - Step 2 takes $O(1)$ time
 - Step 3 takes $O(\log n)$ time because each split takes $O(1)$ time and we perform $O(\log n)$ splits
- ◆ Thus, an insertion in a (2,4) tree takes $O(\log n)$ time

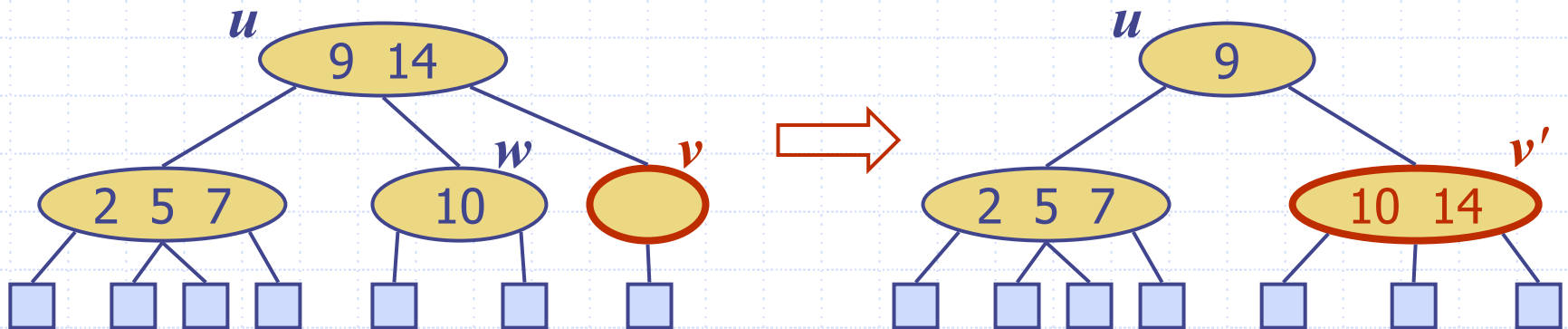
Deletion

- ◆ We reduce deletion of an item to the case where the item is at the node with leaf children
- ◆ Otherwise, we replace the item with its inorder successor (or, equivalently, with its inorder predecessor) and delete the latter item
- ◆ Example: to delete key 24, we replace it with 27 (inorder successor)



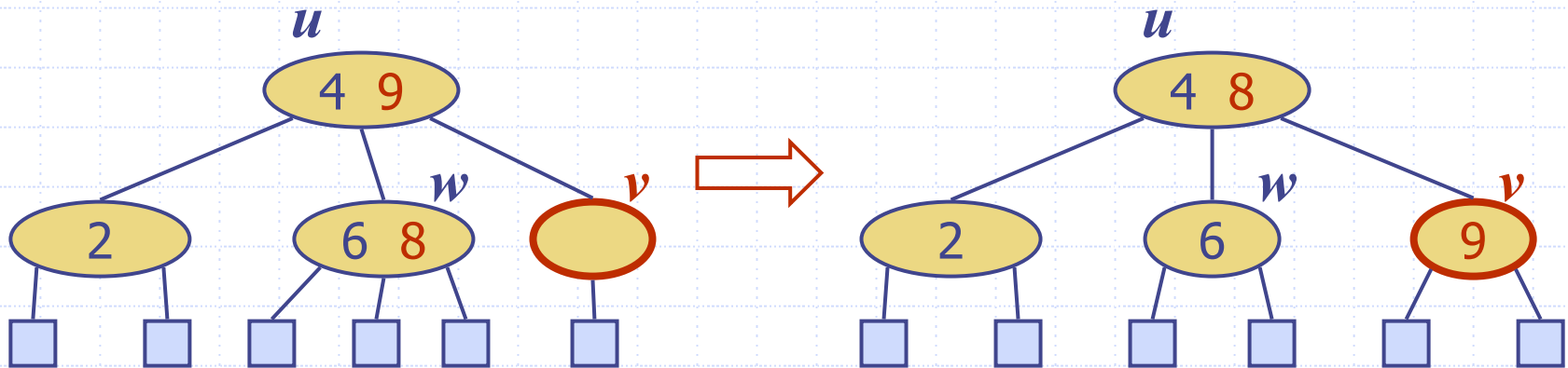
Underflow and Fusion

- ◆ Deleting an item from a node v may cause an **underflow**, where node v becomes a 1-node with one child and no keys
- ◆ To handle an underflow at node v with parent u , we consider two cases
- ◆ **Case 1:** the adjacent siblings of v are 2-nodes
 - **Fusion operation:** we merge v with an adjacent sibling w and move an item from u to the merged node v'
 - After a fusion, the underflow may propagate to the parent u



Underflow and Transfer

- ◆ To handle an underflow at node v with parent u , we consider two cases
- ◆ **Case 2:** an adjacent sibling w of v is a 3-node or a 4-node
 - **Transfer operation:**
 1. we move a child of w to v
 2. we move an item from u to v
 3. we move an item from w to u
 - After a transfer, no underflow occurs



Analysis of Deletion

- ◆ Let T be a $(2,4)$ tree with n items
 - Tree T has $O(\log n)$ height
- ◆ In a deletion operation
 - We visit $O(\log n)$ nodes to locate the node from which to delete the item
 - We handle an underflow with a series of $O(\log n)$ fusions, followed by at most one transfer
 - Each fusion and transfer takes $O(1)$ time
- ◆ Thus, deleting an item from a $(2,4)$ tree takes $O(\log n)$ time

Implementing a Dictionary

◆ Comparison of efficient dictionary implementations

	Search	Insert	Delete	Notes
Hash Table	1 expected	1 expected	1 expected	◆ no ordered dictionary methods ◆ simple to implement
Skip List	$\log n$ high prob.	$\log n$ high prob.	$\log n$ high prob.	◆ randomized insertion ◆ simple to implement
(2,4) Tree	$\log n$ worst-case	$\log n$ worst-case	$\log n$ worst-case	◆ complex to implement