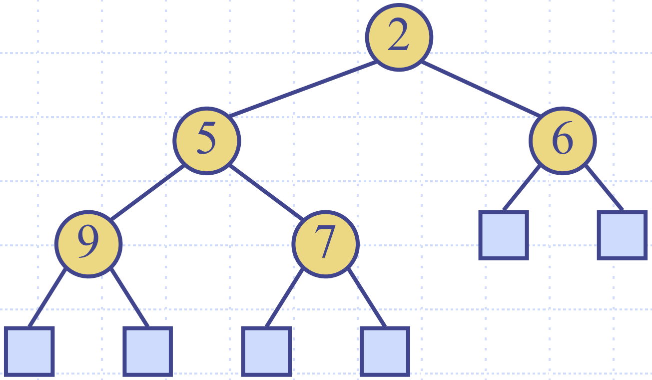
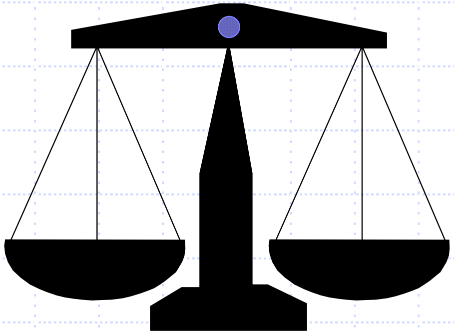
Heaps and Priority Queues



Priority Queue ADT (§7.1)

- A priority queue stores a collection of items
- An item is a pair (key, element)
- Main methods of the Priority Queue ADT
  - insertItem(k, o) inserts an item with key k and element o
  - removeMin() removes the item with the smallest key
- Additional methods
  - minKey(k, o) returns, but does not remove, the smallest key of an item
  - minElement() returns, but does not remove, the element of an item with smallest key
  - size(), isEmpty()
- Applications:
  - Standby flyers
  - Auctions
  - Stock market

Total Order Relation



- Keys in a priority queue can be arbitrary objects on which an order is defined
- Two distinct items in a priority queue can have the same key
- Mathematical concept of total order relation $\leq$
  - **Reflexive** property: $x \leq x$
  - **Antisymmetric** property: $x \leq y \wedge y \leq x \Rightarrow x = y$
  - **Transitive** property: $x \leq y \wedge y \leq z \Rightarrow x \leq z$

Comparator ADT (§7.1.4)

- A *comparator* encapsulates the action of comparing two objects according to a given total order relation
- A generic priority queue uses a comparator as a template argument, to define the comparison function (<,=,>)
- The comparator is external to the keys being compared. Thus, the same objects can be sorted in different ways by using different comparators.
- When the priority queue needs to compare two keys, it uses its comparator

Using Comparators in C++

- A comparator class overloads the "()" operator with a comparison function.
- Example: Compare two points in the plane lexicographically.

```
class LexCompare {
public:
    int operator()(Point a, Point b) {
        if      (a.x < b.x) return –1
        else if (a.x > b.x) return +1
        else if (a.y < b.y) return  –1
        else if (a.y > b.y) return +1
        else return 0;
    }
};
```

- To use the comparator, define an object of this type, and invoke it using its "()" operator:
- Example of usage:

```
Point p(2.3, 4.5);
Point q(1.7, 7.3);
LexCompare lexCompare;

if (lexCompare(p, q) < 0)
    cout << "p less than q";
else if (lexCompare(p, q) == 0)
    cout << "p equals q";
else if (lexCompare(p, q) > 0)
    cout << "p greater than q";
```

Sorting with a Priority Queue (§7.1.2)

- We can use a priority queue to sort a set of comparable elements
  - Insert the elements one by one with a series of insertItem(e, e) operations
  - Remove the elements in sorted order with a series of removeMin() operations
- The running time of this sorting method depends on the priority queue implementation

Algorithm *PQ-Sort*(*S*, *C*)
    Input sequence ***S***, comparator ***C*** for the elements of ***S***
    Output sequence ***S*** sorted in increasing order according to ***C***
    P ← priority queue with comparator ***C***
    while !*S.isEmpty* ()
        e ← ***S.remove*** (***S. first*** ())
        P.insertItem(*e*, *e*)
    while !*P.isEmpty*()
        e ← ***P.minElement***()
        P.removeMin()
        S.insertLast(*e*)

Sequence-based Priority Queue

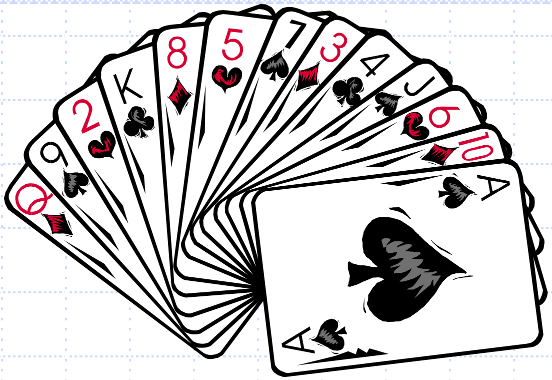
- Implementation with an unsorted list

- Performance:
  - insertItem takes $O(1)$ time since we can insert the item at the beginning or end of the sequence
  - removeMin, minKey and minElement take $O(n)$ time since we have to traverse the entire sequence to find the smallest key

- Implementation with a sorted list

- Performance:
  - insertItem takes $O(n)$ time since we have to find the place where to insert the item
  - removeMin, minKey and minElement take $O(1)$ time since the smallest key is at the beginning of the sequence

Selection-Sort



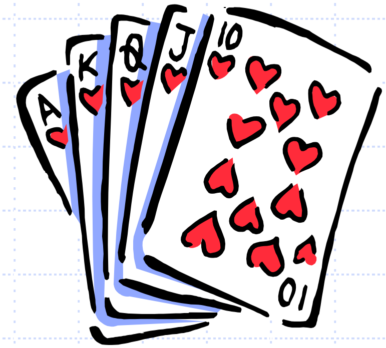
- Selection-sort is the variation of PQ-sort where the priority queue is implemented with an unsorted sequence

- Running time of Selection-sort:
  - Inserting the elements into the priority queue with n insertItem operations takes $O(n)$ time
  - Removing the elements in sorted order from the priority queue with n removeMin operations takes time proportional to

$$1 + 2 + \ldots + n$$

- Selection-sort runs in $O(n^2)$ time

Insertion-Sort



- Insertion-sort is the variation of PQ-sort where the priority queue is implemented with a sorted sequence

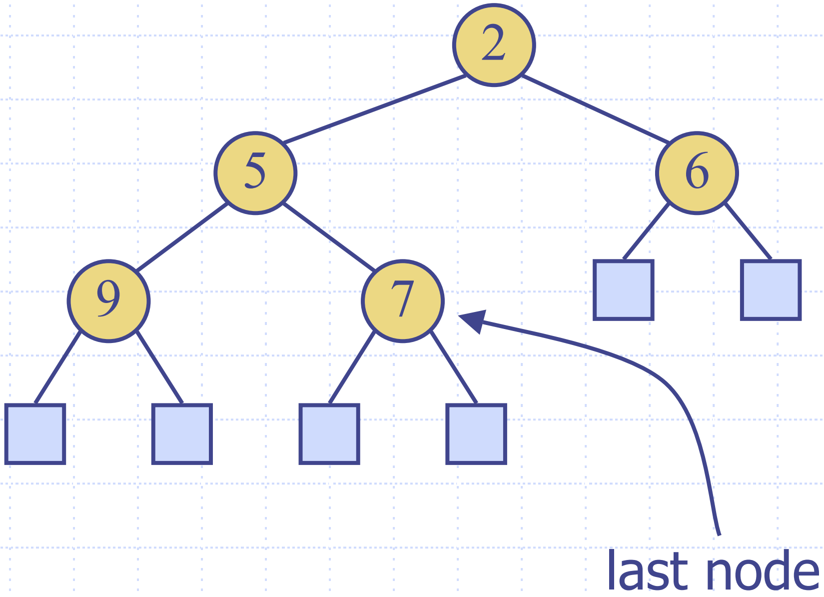
- Running time of Insertion-sort:
  - Inserting the elements into the priority queue with n insertItem operations takes time proportional to

    $$1 + 2 + \ldots + n$$

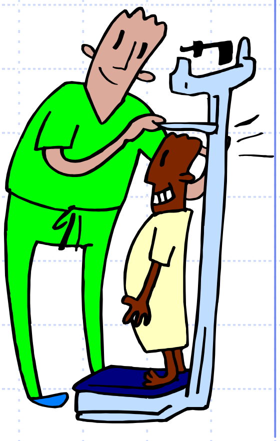
  - Removing the elements in sorted order from the priority queue with a series of n removeMin operations takes $O(n)$ time
- Insertion-sort runs in $O(n^2)$ time

What is a heap? (§7.3.1)

- A heap is a binary tree storing keys at its internal nodes and satisfying the following properties:
  - Heap-Order: for every internal node v other than the root, $key(v) \geq key(parent(v))$
  - Complete Binary Tree: let h be the height of the heap
    - for $i = 0, \ldots, h-1$, there are 2^i nodes of depth i
    - at depth $h-1$, the internal nodes are to the left of the external nodes
- The last node of a heap is the rightmost internal node of depth $h-1$

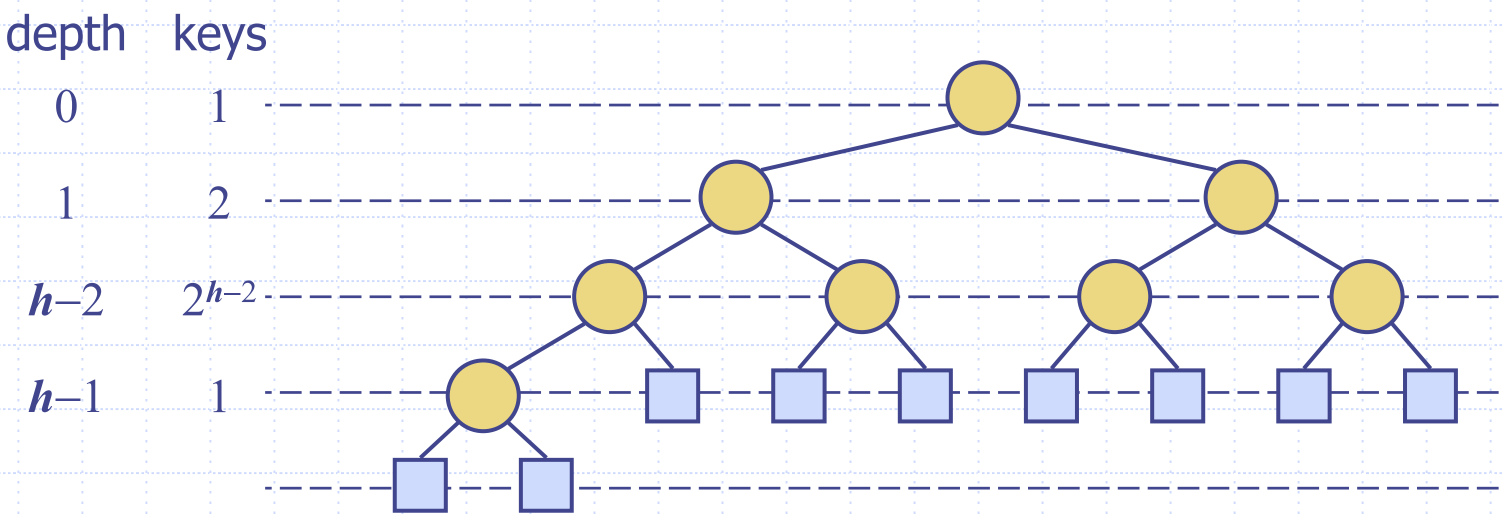


Height of a Heap



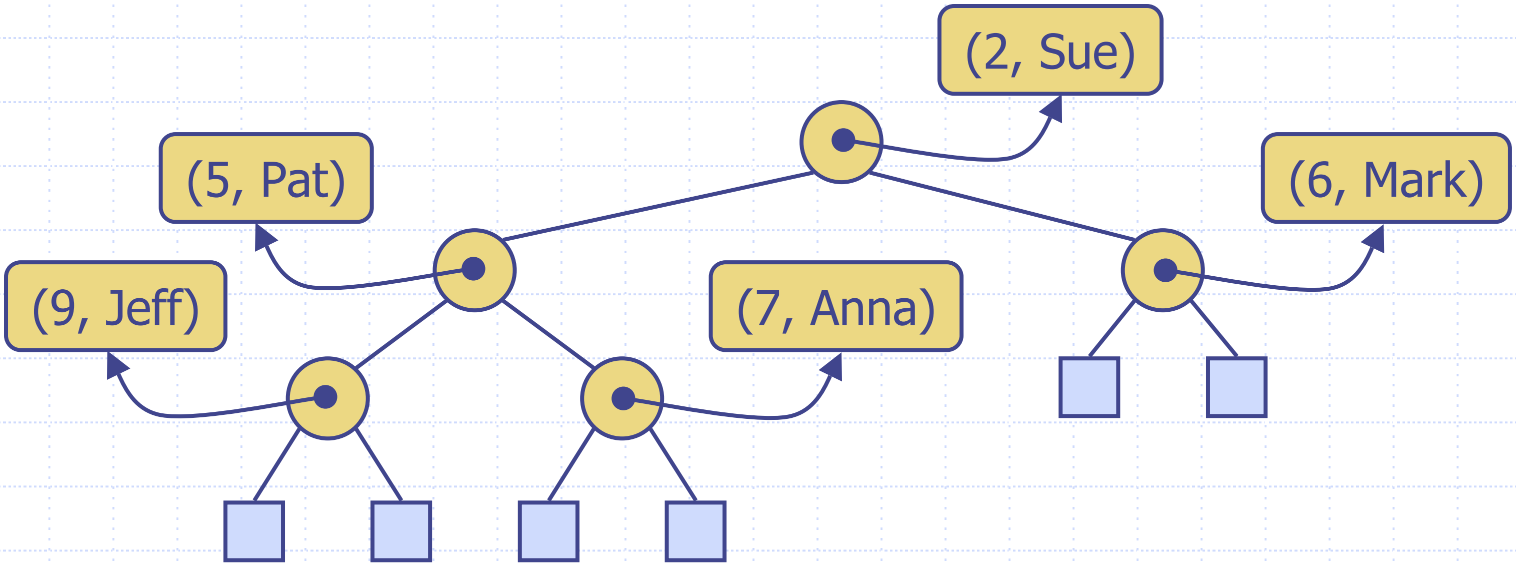
- **Theorem:** A heap storing n keys has height $O(\log n)$

  Proof: (we apply the complete binary tree property)
  - Let h be the height of a heap storing n keys
  - Since there are 2^i keys at depth $i = 0, \dots, h-2$ and at least one key at depth $h-1$, we have $n \geq 1 + 2 + 4 + \dots + 2^{h-2} + 1$
  - Thus, $n \geq 2^{h-1}$, i.e., $h \leq \log n + 1$



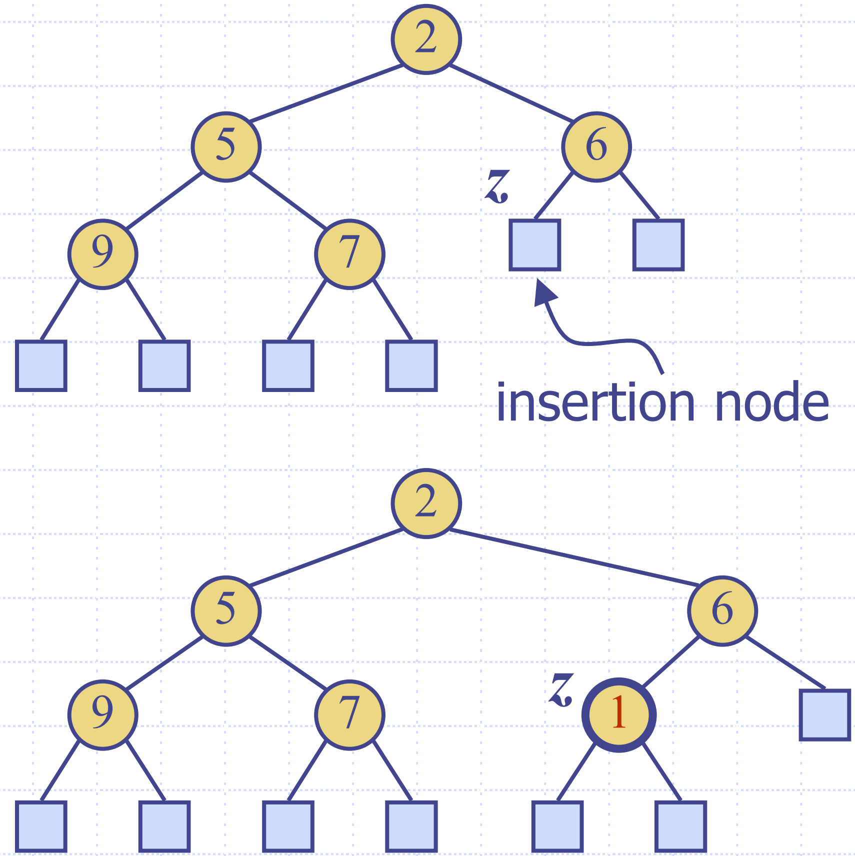
Heaps and Priority Queues

- We can use a heap to implement a priority queue
- We store a (key, element) item at each internal node
- We keep track of the position of the last node
- For simplicity, we show only the keys in the pictures



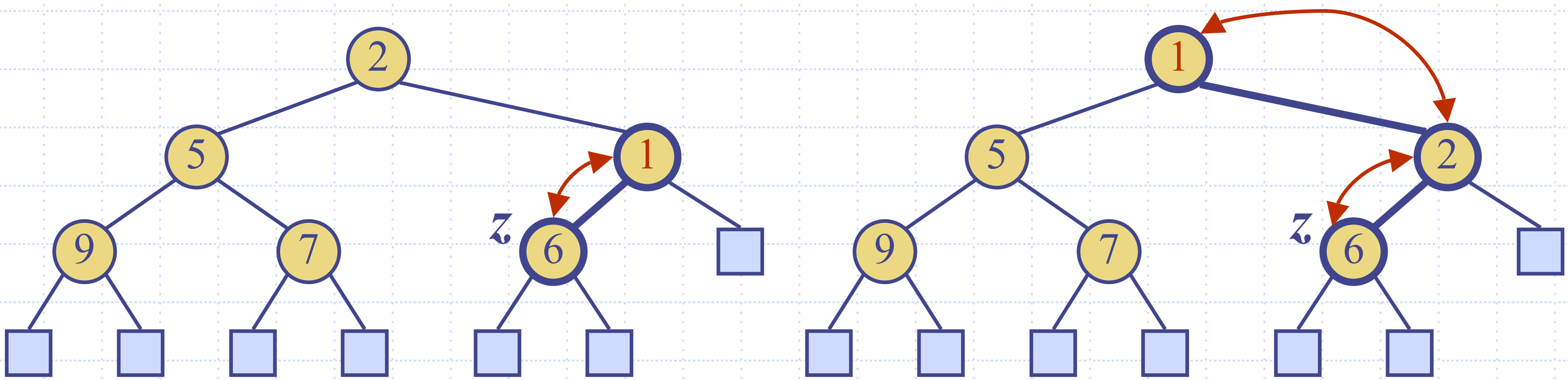
Insertion into a Heap (§7.3.2)

- Method insertItem of the priority queue ADT corresponds to the insertion of a key k to the heap
- The insertion algorithm consists of three steps
  - Find the insertion node z (the new last node)
  - Store k at z and expand z into an internal node
  - Restore the heap-order property (discussed next)



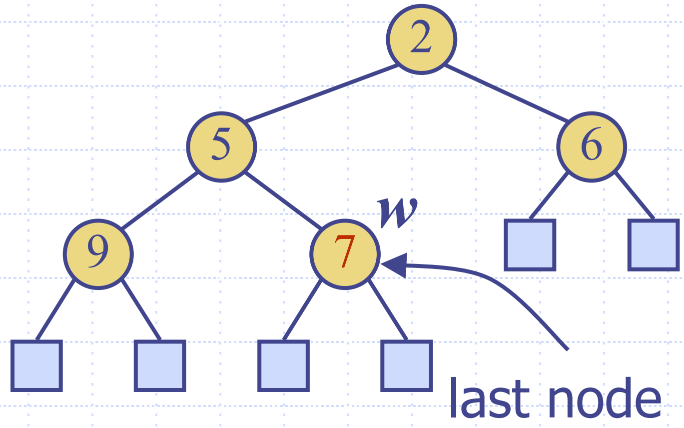
Upheap

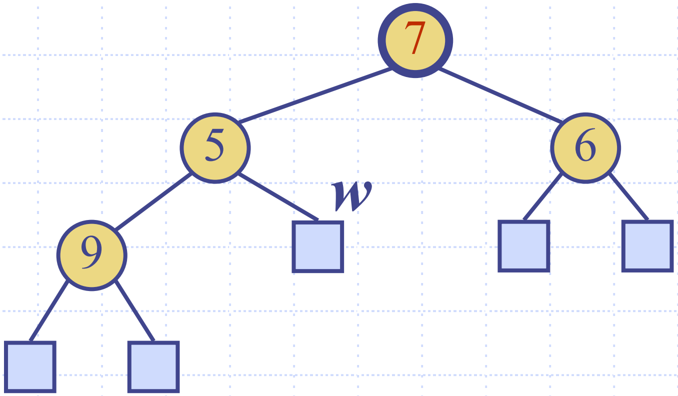
- After the insertion of a new key k, the heap-order property may be violated
- Algorithm upheap restores the heap-order property by swapping k along an upward path from the insertion node
- Upheap terminates when the key k reaches the root or a node whose parent has a key smaller than or equal to k
- Since a heap has height $O(\log n)$, upheap runs in $O(\log n)$ time



Removal from a Heap (§7.3.2)

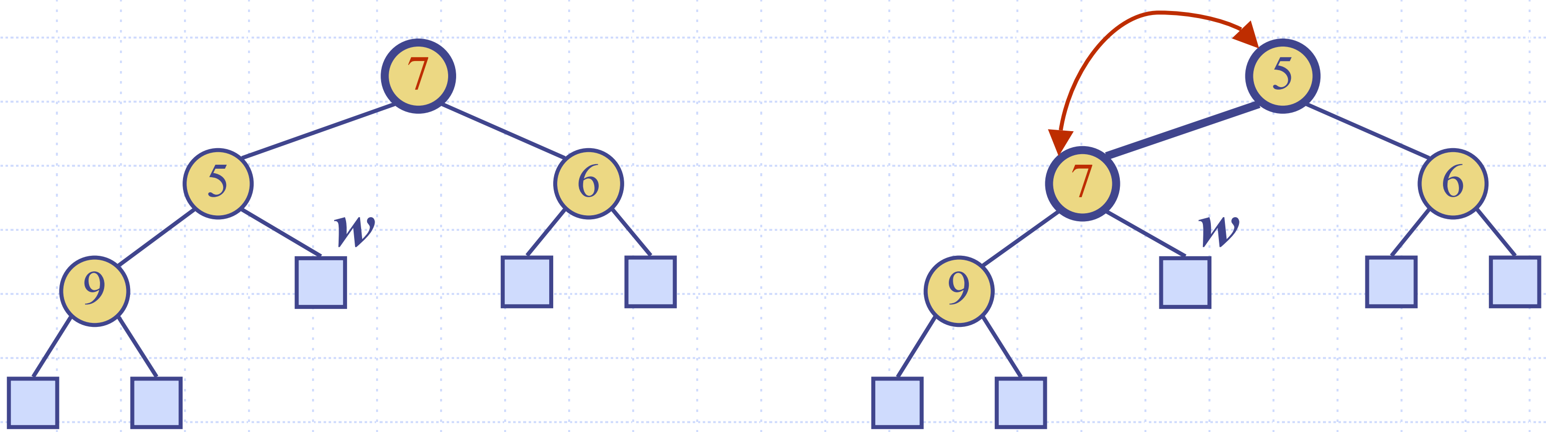
- Method removeMin of the priority queue ADT corresponds to the removal of the root key from the heap
- The removal algorithm consists of three steps
  - Replace the root key with the key of the last node *w*
  - Compress *w* and its children into a leaf
  - Restore the heap-order property (discussed next)





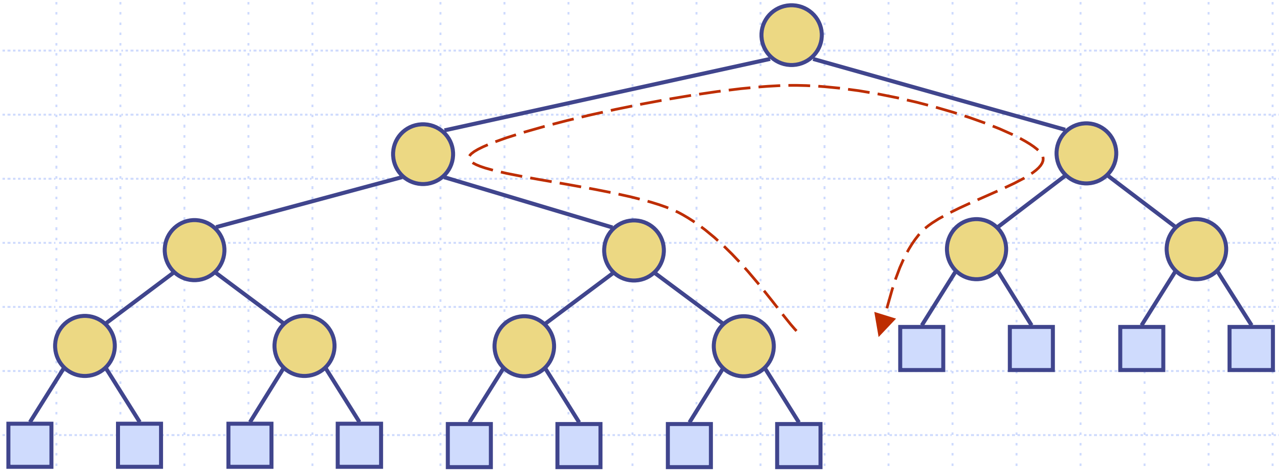
Downheap

- After replacing the root key with the key k of the last node, the heap-order property may be violated
- Algorithm downheap restores the heap-order property by swapping key k along a downward path from the root
- Upheap terminates when key k reaches a leaf or a node whose children have keys greater than or equal to k
- Since a heap has height $O(\log n)$, downheap runs in $O(\log n)$ time

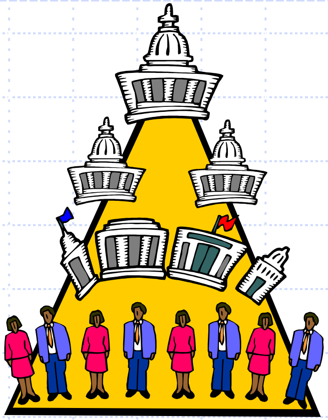


Updating the Last Node

- The insertion node can be found by traversing a path of $O(\log n)$ nodes
  - Go up until a left child or the root is reached
  - If a left child is reached, go to the right child
  - Go down left until a leaf is reached
- Similar algorithm for updating the last node after a removal



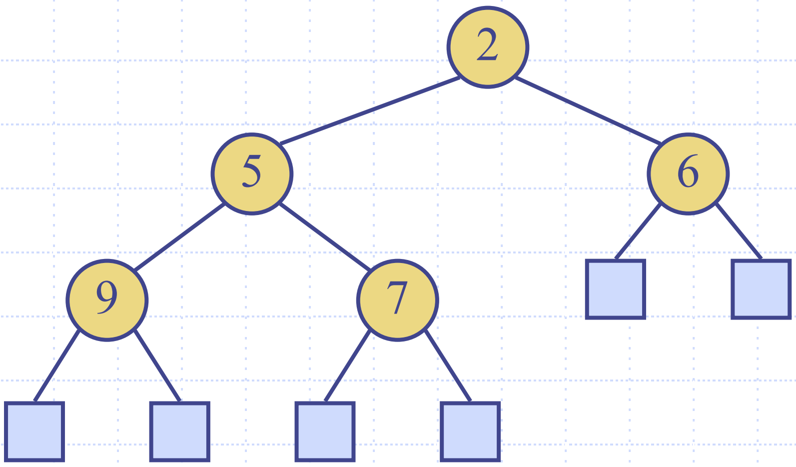
Heap-Sort (§7.3.4)

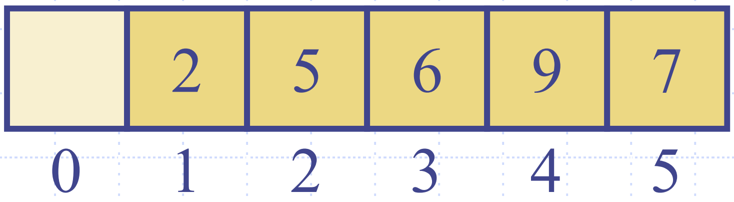


- Consider a priority queue with n items implemented by means of a heap
  - the space used is $O(n)$
  - methods insertItem and removeMin take $O(\log n)$ time
  - methods size, isEmpty, minKey, and minElement take time $O(1)$ time
- Using a heap-based priority queue, we can sort a sequence of n elements in $O(n \log n)$ time
- The resulting algorithm is called heap-sort
- Heap-sort is much faster than quadratic sorting algorithms, such as insertion-sort and selection-sort

Vector-based Heap Implementation (§7.3.3)

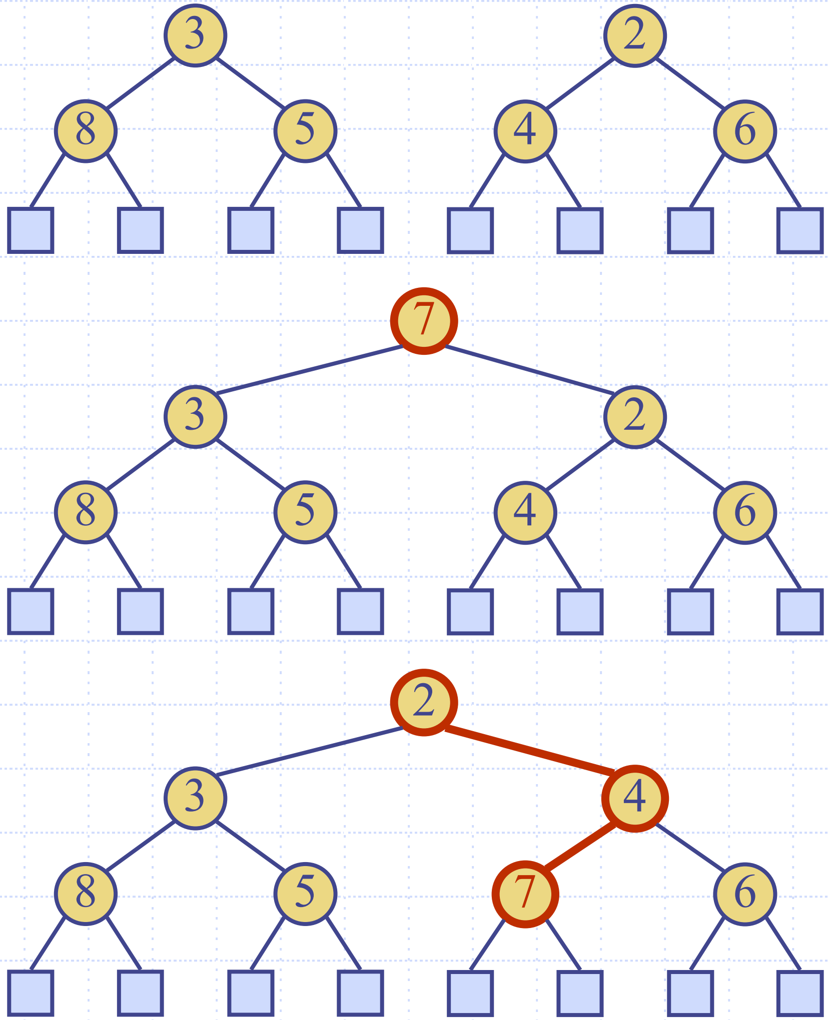
- We can represent a heap with n keys by means of a vector of length $n + 1$
- For the node at rank i
  - the left child is at rank $2i$
  - the right child is at rank $2i + 1$
- Links between nodes are not explicitly stored
- The leaves are not represented
- The cell of at rank 0 is not used
- Operation insertItem corresponds to inserting at rank $n + 1$
- Operation removeMin corresponds to removing at rank n
- Yields in-place heap-sort





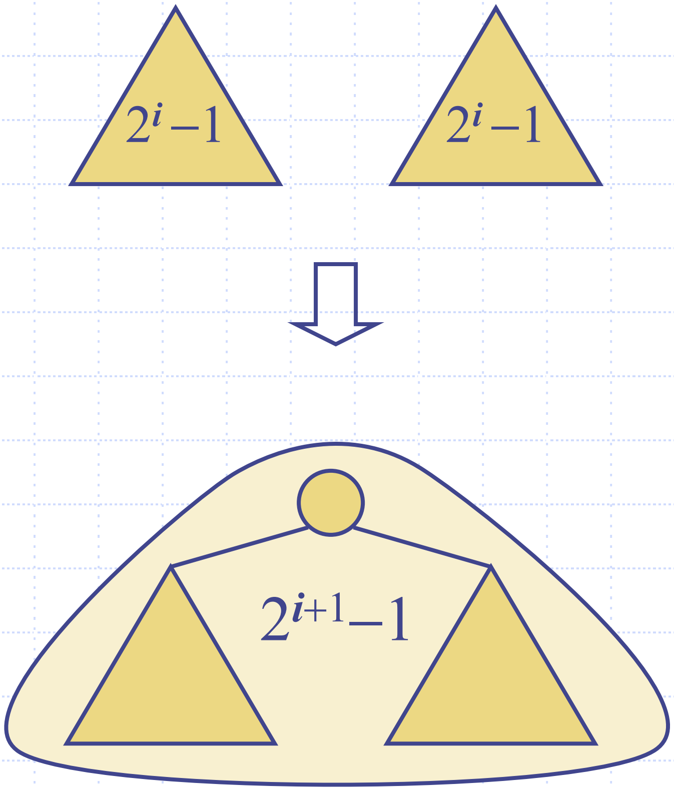
Merging Two Heaps

- We are given two two heaps and a key k
- We create a new heap with the root node storing k and with the two heaps as subtrees
- We perform downheap to restore the heap-order property

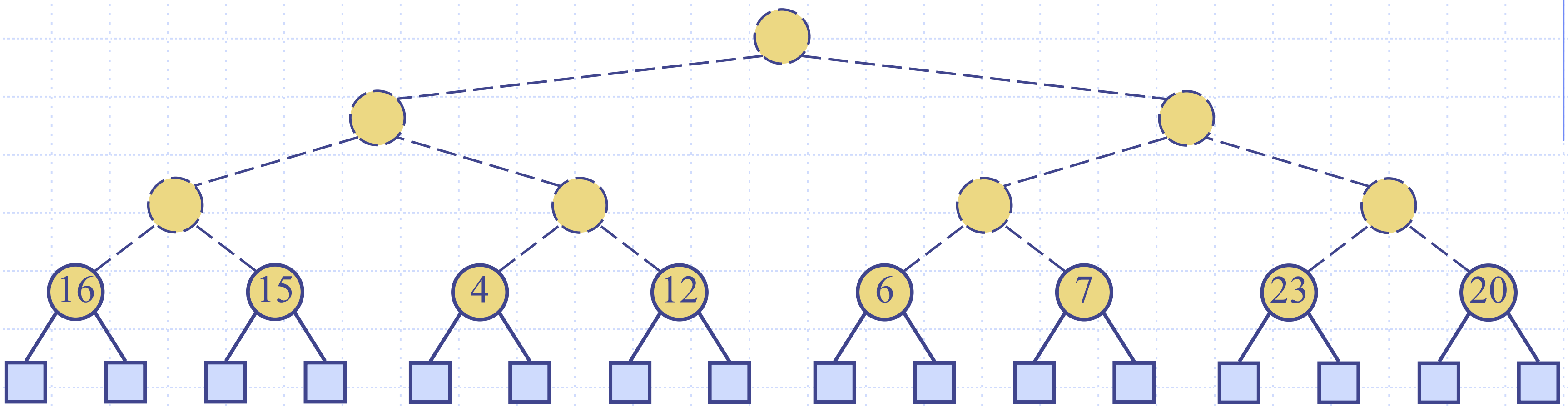


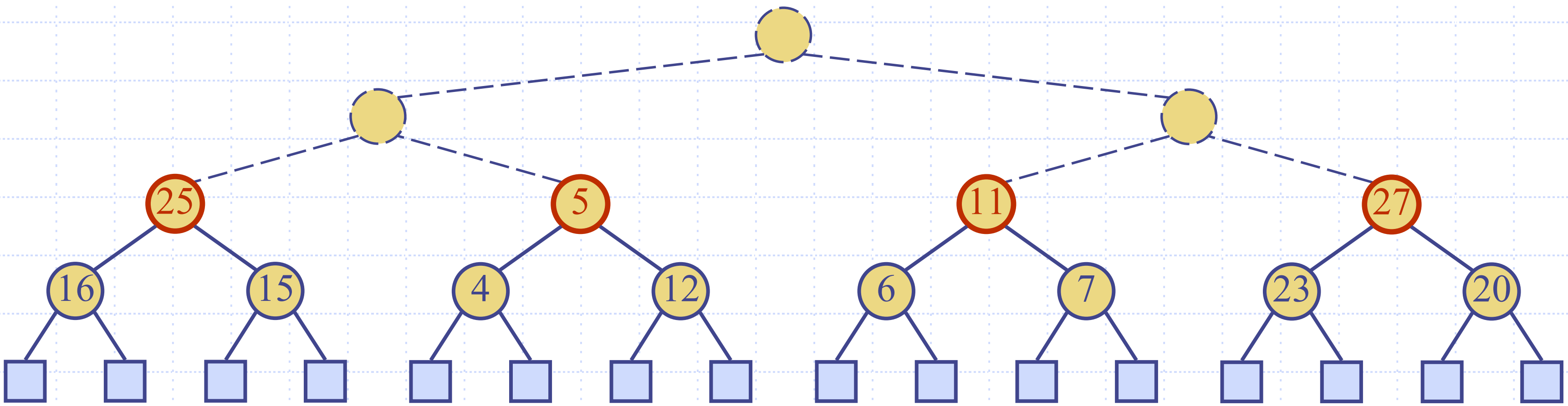
Bottom-up Heap Construction (§7.3.5)

- We can construct a heap storing n given keys in using a bottom-up construction with $\log n$ phases
- In phase i, pairs of heaps with $2^i - 1$ keys are merged into heaps with $2^{i+1} - 1$ keys

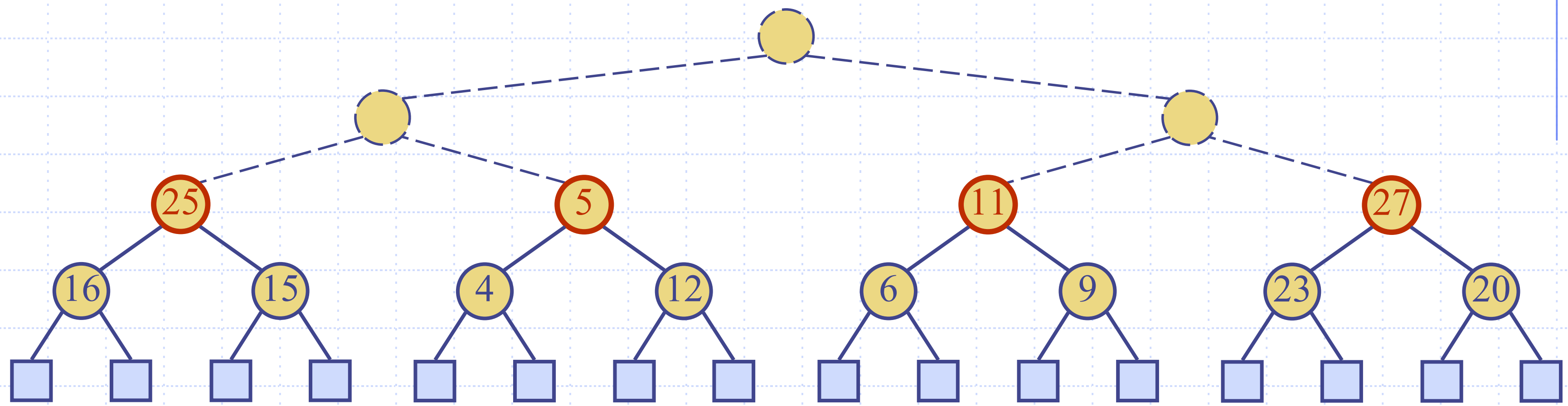


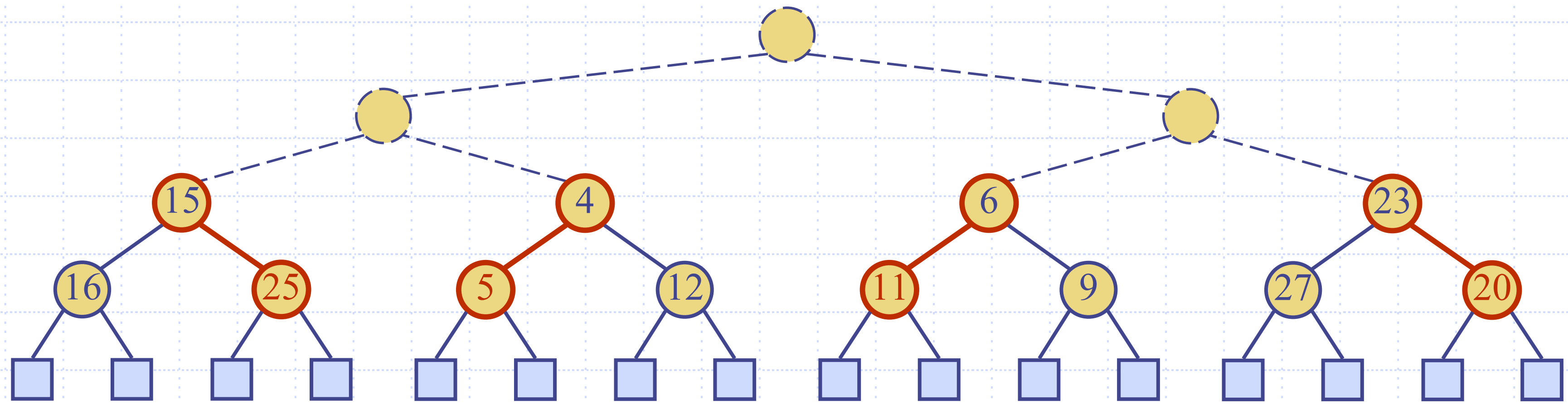
Example



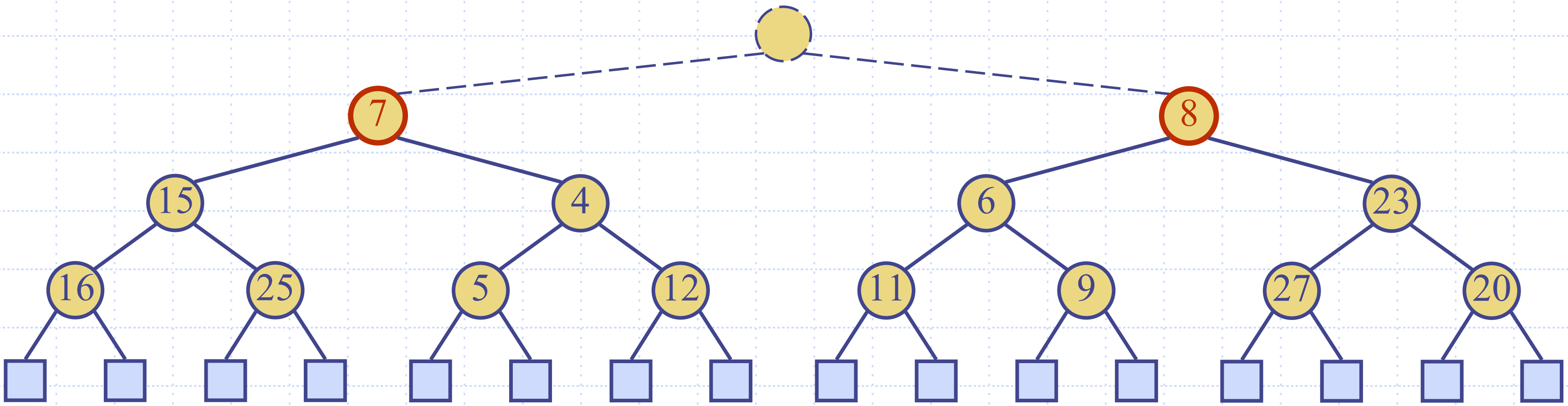


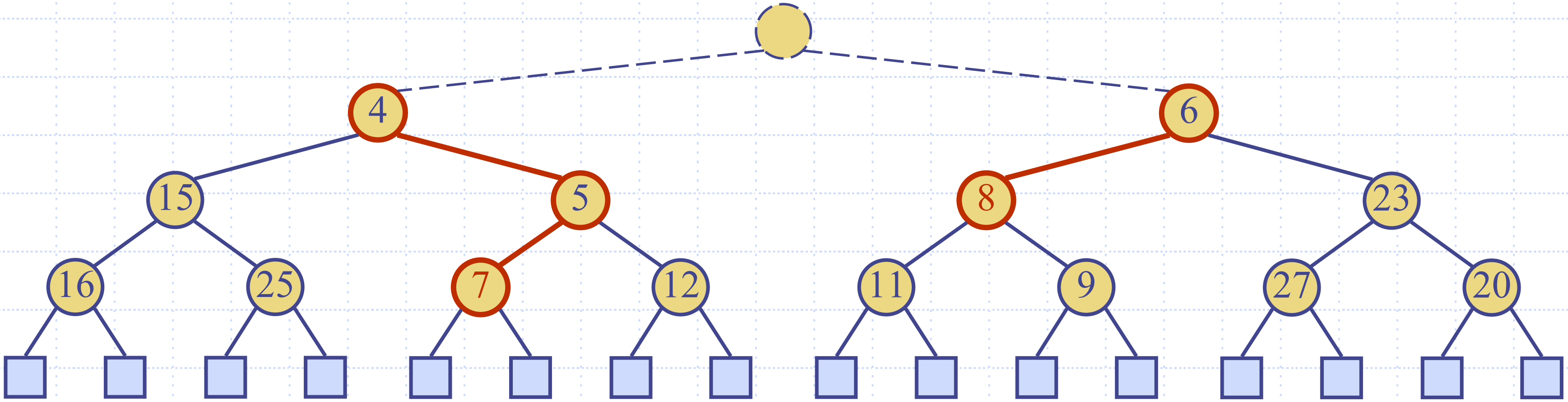
Example (contd.)



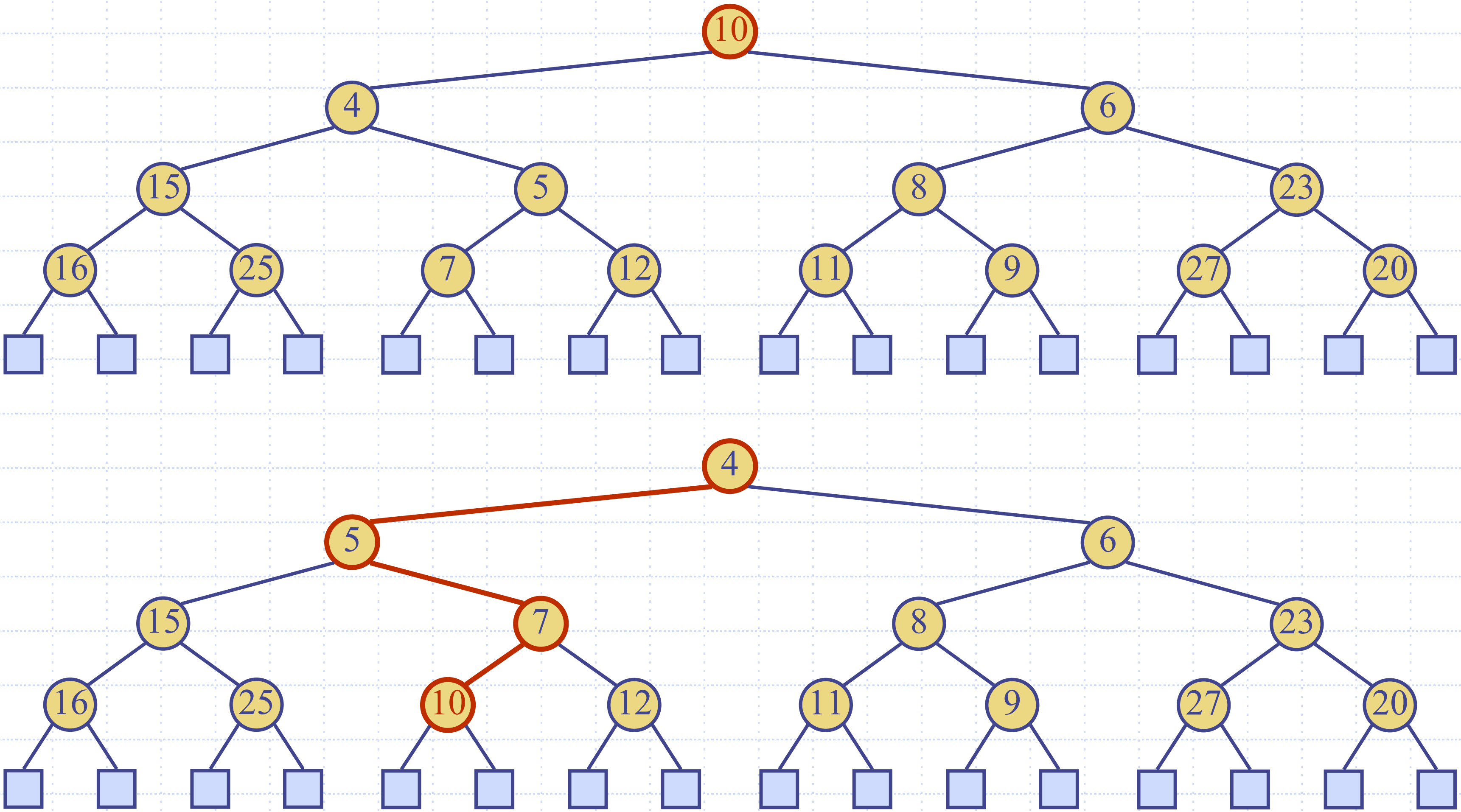


Example (contd.)





Example (end)



Analysis

- We visualize the worst-case time of a downheap with a proxy path that goes first right and then repeatedly goes left until the bottom of the heap (this path may differ from the actual downheap path)
- Since each node is traversed by at most two proxy paths, the total number of nodes of the proxy paths is $O(n)$
- Thus, bottom-up heap construction runs in $O(n)$ time
- Bottom-up heap construction is faster than n successive insertions and speeds up the first phase of heap-sort