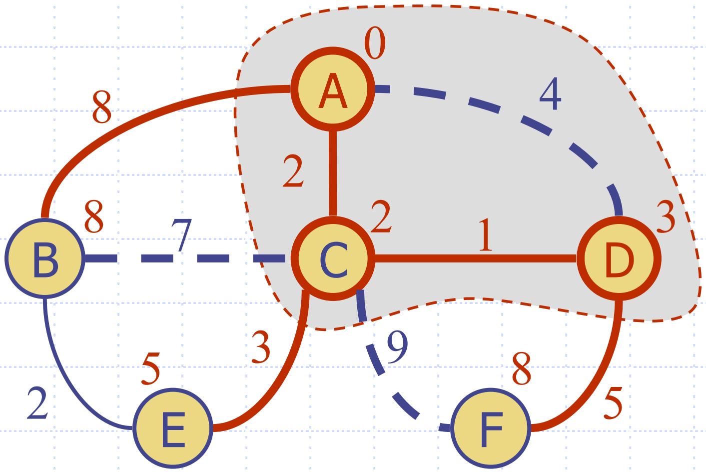
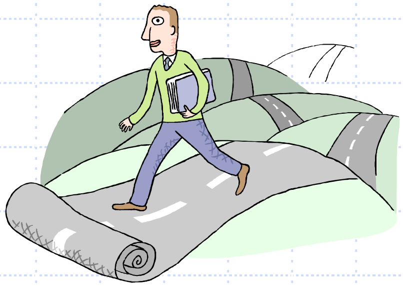
Shortest Paths



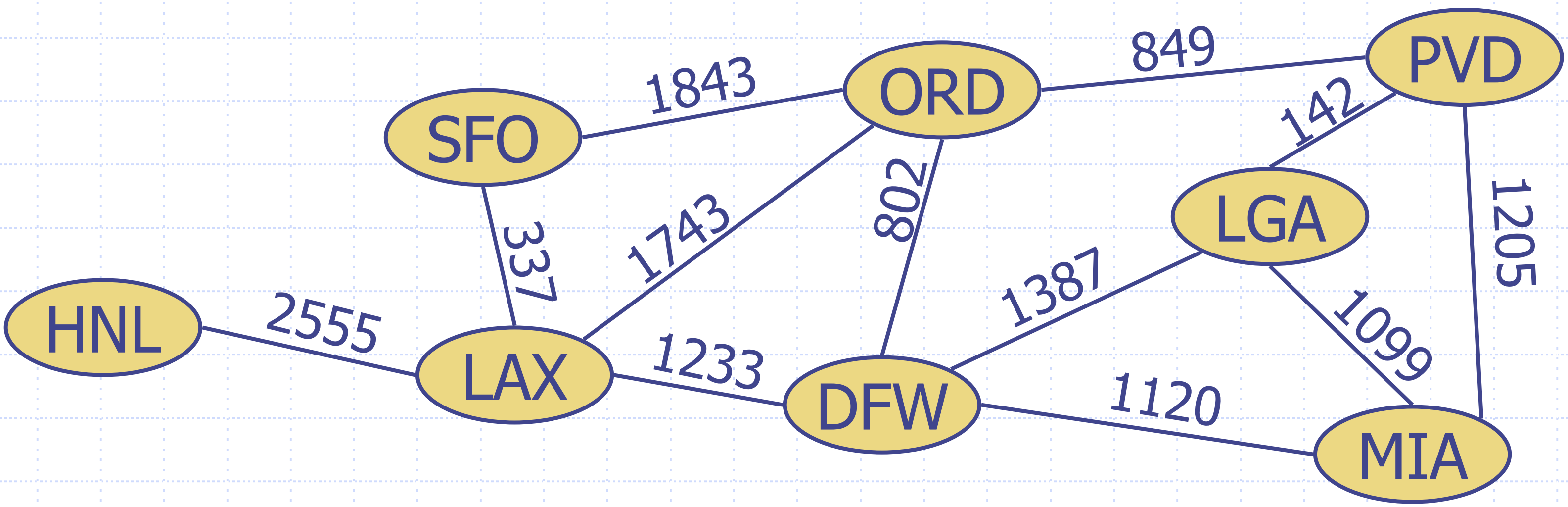
Outline and Reading

- Weighted graphs (§12.1)
  - Shortest path problem
  - Shortest path properties
- Dijkstra's algorithm (§12.6.1)
  - Algorithm
  - Edge relaxation
- The Bellman-Ford algorithm
- Shortest paths in DAGs
- All-pairs shortest paths

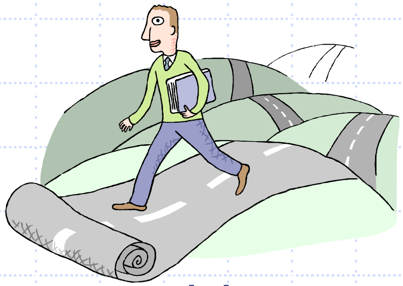
Weighted Graphs



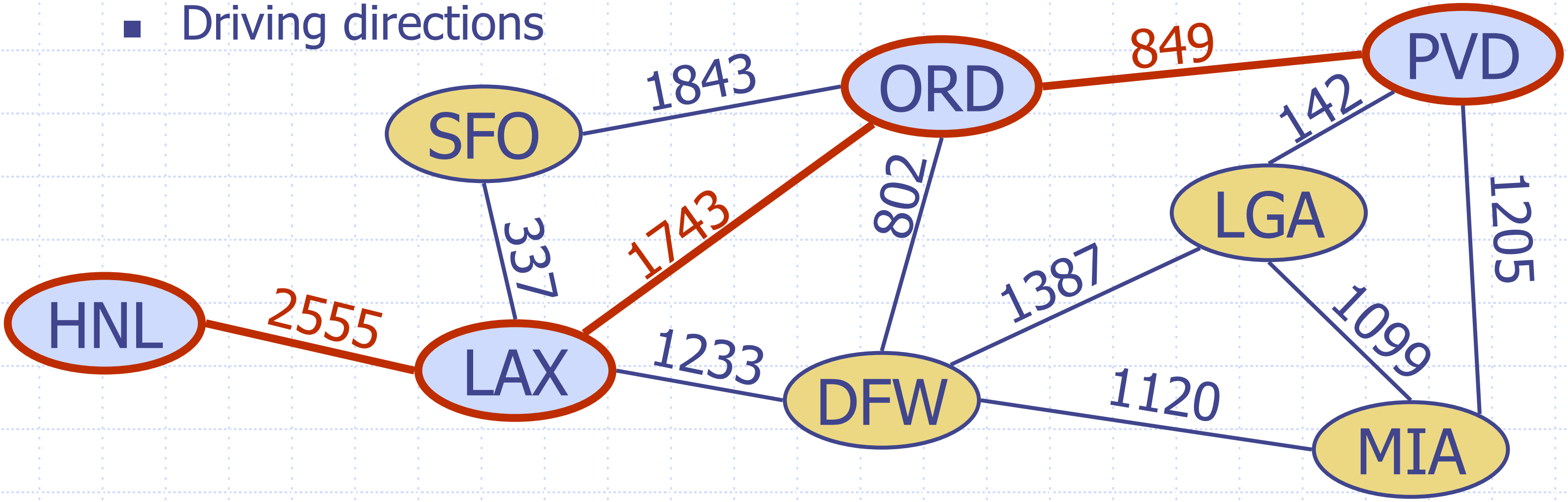
- In a weighted graph, each edge has an associated numerical value, called the weight of the edge
- Edge weights may represent, distances, costs, etc.
- Example:
  - In a flight route graph, the weight of an edge represents the distance in miles between the endpoint airports



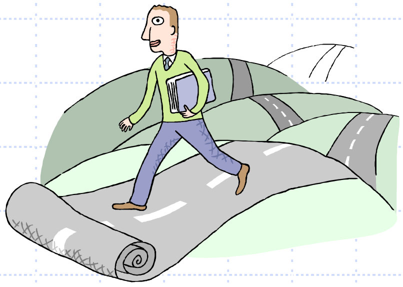
Shortest Path Problem



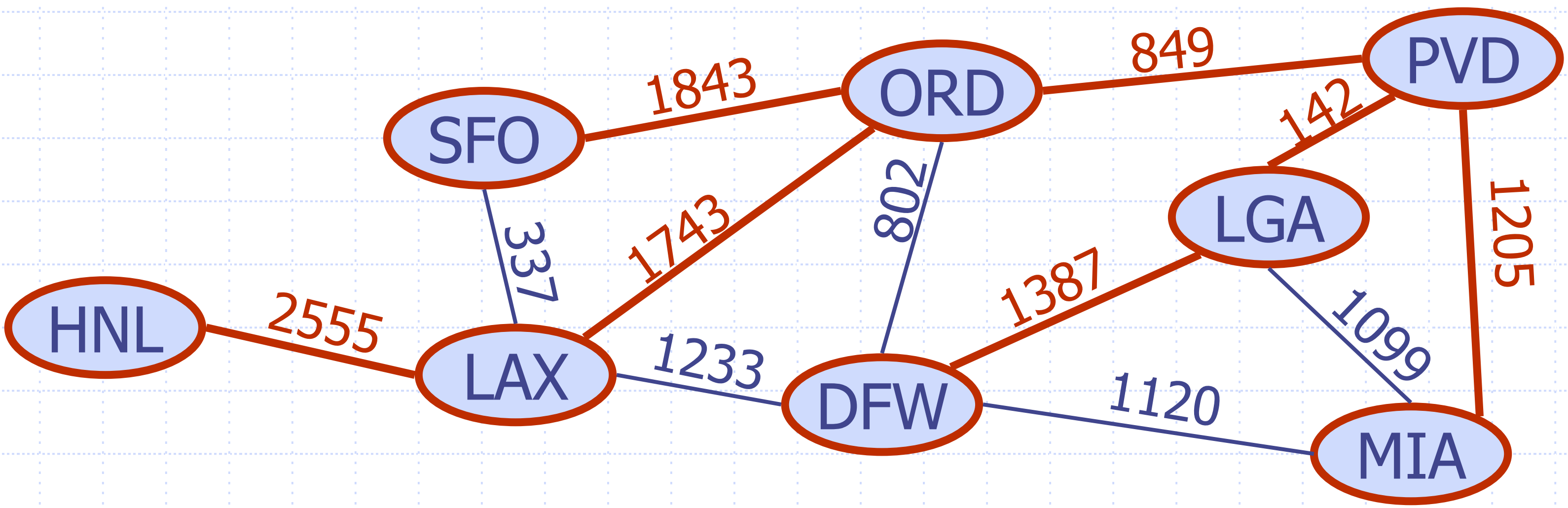
- Given a weighted graph and two vertices u and v, we want to find a path of minimum total weight between u and v.
  - Length of a path is the sum of the weights of its edges.
- Example:
  - Shortest path between Providence and Honolulu
- Applications
  - Internet packet routing
  - Flight reservations
  - Driving directions



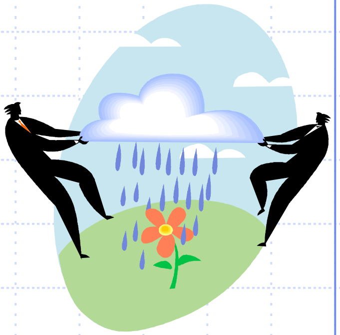
Shortest Path Properties



- Property 1:
  - A subpath of a shortest path is itself a shortest path
- Property 2:
  - There is a tree of shortest paths from a start vertex to all the other vertices
- Example:
  - Tree of shortest paths from Providence



Dijkstra’s Algorithm

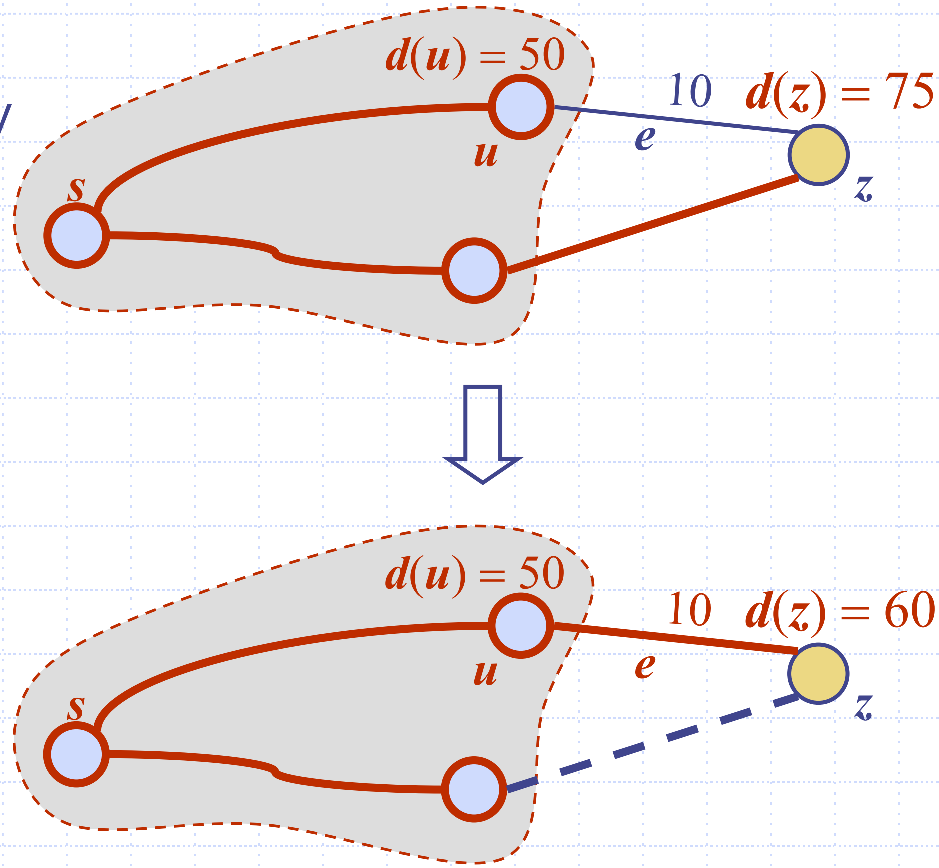


- The distance of a vertex v from a vertex s is the length of a shortest path between s and v
- Dijkstra’s algorithm computes the distances of all the vertices from a given start vertex s
- Assumptions:
  - the graph is connected
  - the edges are undirected
  - the edge weights are nonnegative
- We grow a “**cloud**” of vertices, beginning with s and eventually covering all the vertices
- We store with each vertex v a label $d(v)$ representing the distance of v from s in the subgraph consisting of the cloud and its adjacent vertices
- At each step
  - We add to the cloud the vertex u outside the cloud with the smallest distance label, $d(u)$
  - We update the labels of the vertices adjacent to u

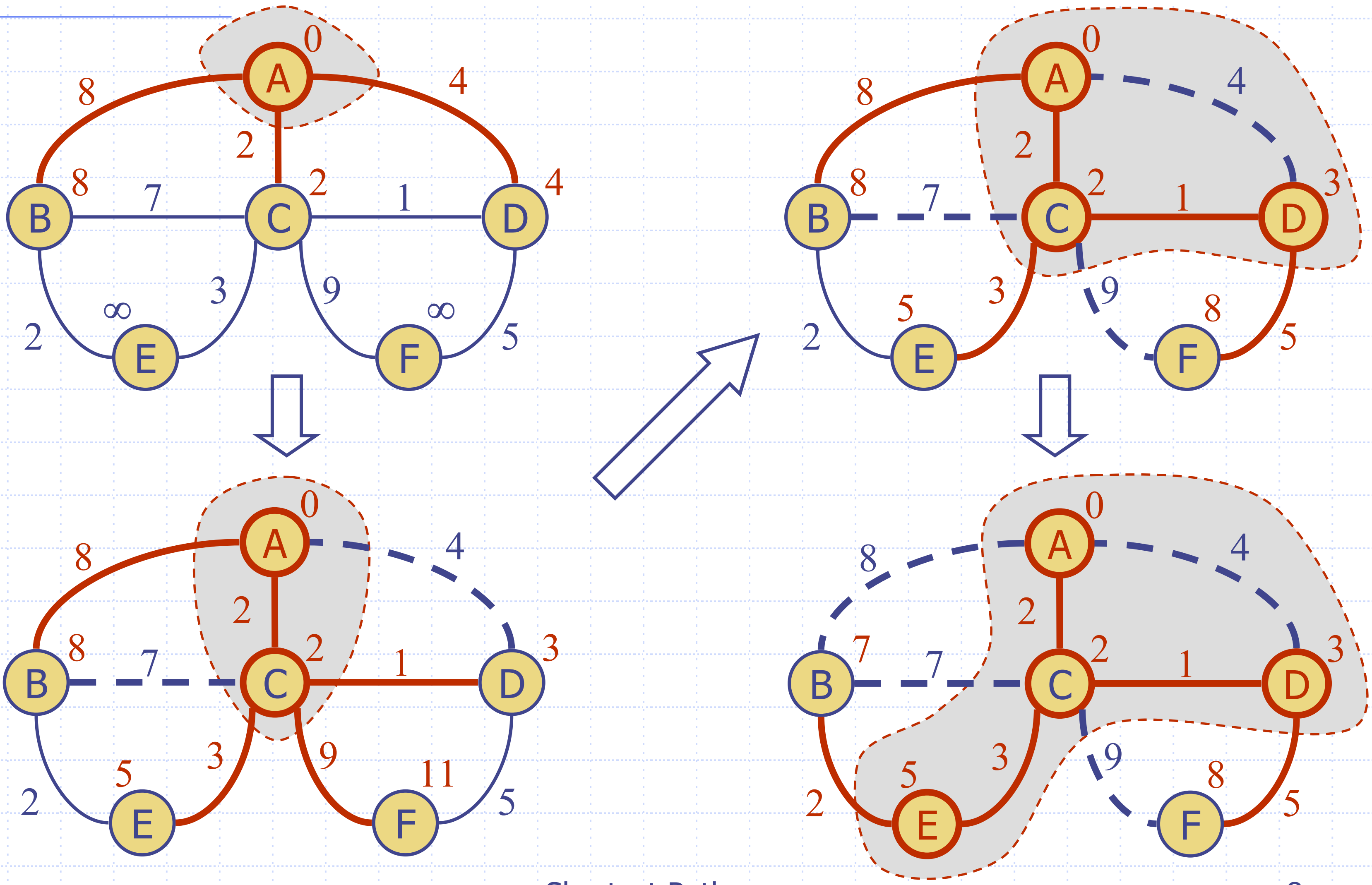
Edge Relaxation

- Consider an edge $e = (u,z)$ such that
  - u is the vertex most recently added to the cloud
  - z is not in the cloud
- The relaxation of edge e updates distance $d(z)$ as follows:

$$d(z) \leftarrow \min\{d(z), d(u) + weight(e)\}$$

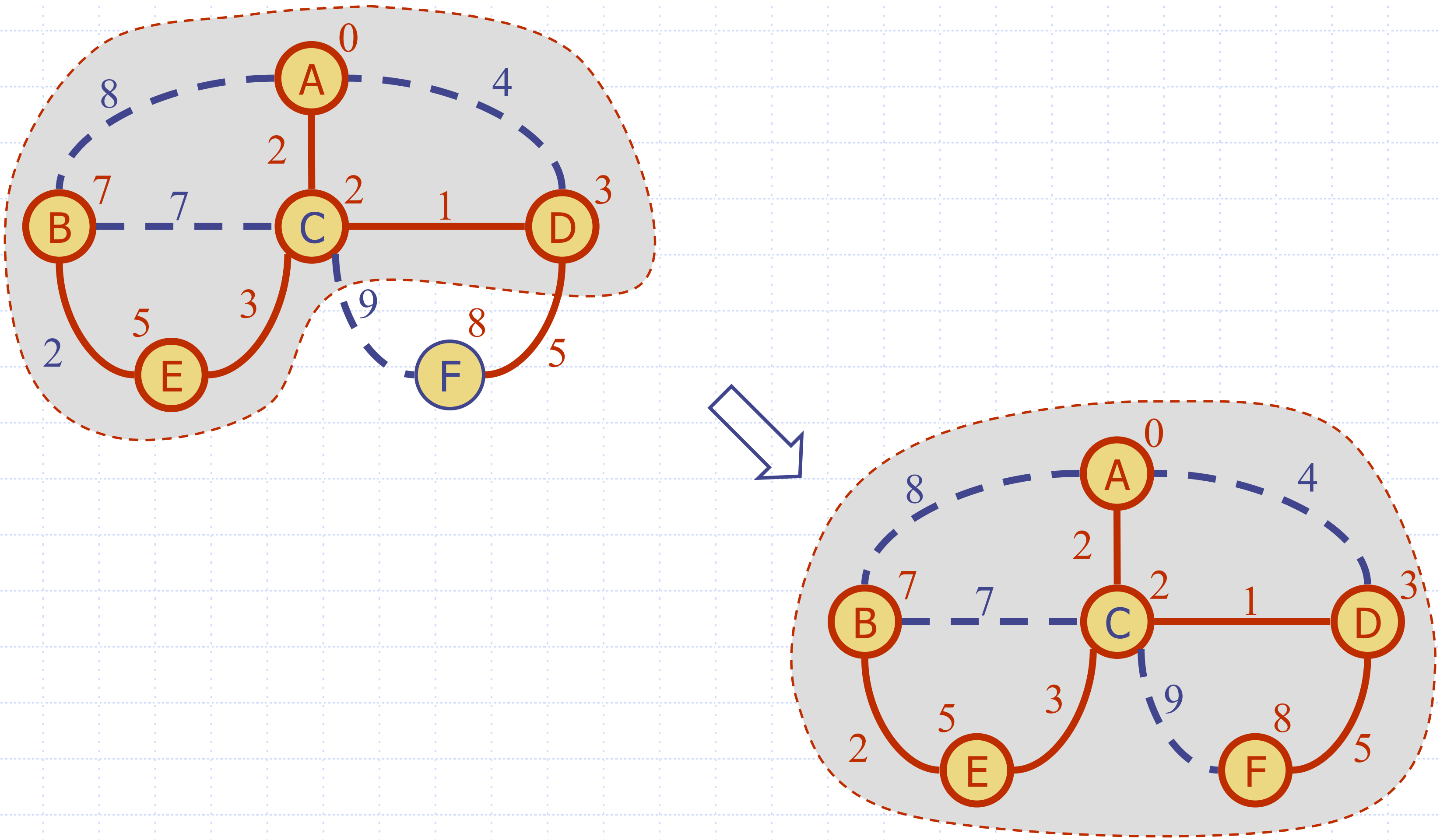


Example

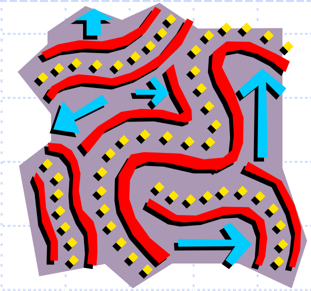


A B C D E F
0 8 4 2 8 2 4 7 1 ∞ ∞ 2 3 9 5
0 8 4 2 8 2 3 7 1 5 11 2 3 9 5
0 8 4 2 8 2 3 7 1 5 8 2 3 9 5
0 8 4 2 7 2 3 7 1 5 8 2 3 9 5

Example (cont.)



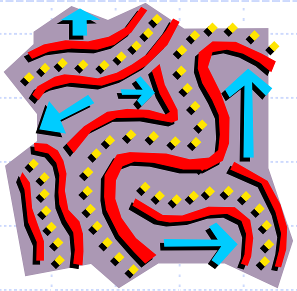
Dijkstra's Algorithm



- A priority queue stores the vertices outside the cloud
  - Key: distance
  - Element: vertex
- Locator-based methods
  - *insert*(*k*,*e*) returns a locator
  - *replaceKey*(*l*,*k*) changes the key of an item
- We store two labels with each vertex:
  - distance (d(v) label)
  - locator in priority queue

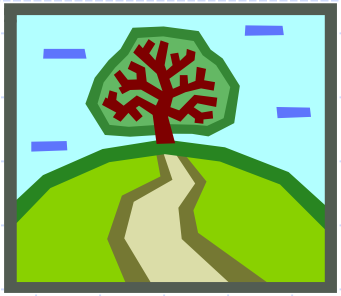
```
Algorithm DijkstraDistances(G, s)
    Q ← new heap-based priority queue
    for all  v ∈ G.vertices()
        if  v = s
            setDistance(v, 0)
        else
            setDistance(v, ∞)
        l ← Q.insert(getDistance(v), v)
        setLocator(v,l)
    while  ¬Q.isEmpty()
        u ← Q.removeMin()
        for all  e ∈ G.incidentEdges(u)
            { relax edge e }
            z ← G.opposite(u,e)
            r ← getDistance(u) + weight(e)
            if  r < getDistance(z)
                setDistance(z,r)
                Q.replaceKey(getLocator(z),r)
```

Analysis



- Graph operations
  - Method incidentEdges is called once for each vertex
- Label operations
  - We set/get the distance and locator labels of vertex z $O(\deg(z))$ times
  - Setting/getting a label takes $O(1)$ time
- Priority queue operations
  - Each vertex is inserted once into and removed once from the priority queue, where each insertion or removal takes $O(\log n)$ time
  - The key of a vertex in the priority queue is modified at most $\deg(w)$ times, where each key change takes $O(\log n)$ time
- Dijkstra's algorithm runs in $O((n + m) \log n)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$
- The running time can also be expressed as $O(m \log n)$ since the graph is connected

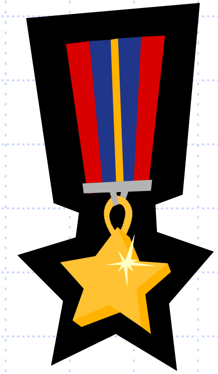
Extension



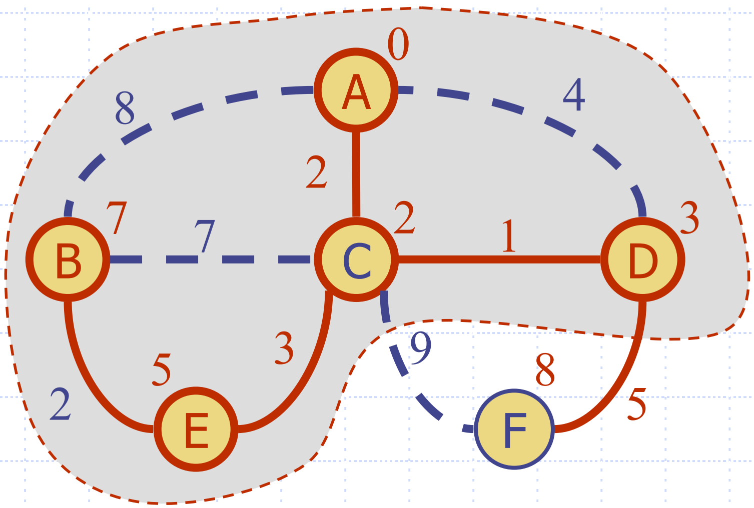
- Using the template method pattern, we can extend Dijkstra's algorithm to return a tree of shortest paths from the start vertex to all other vertices
- We store with each vertex a third label:
  - parent edge in the shortest path tree
- In the edge relaxation step, we update the parent label

```
Algorithm DijkstraShortestPathsTree(G, s)

    …

  for all v ∈ G.vertices()
    …
    setParent(v, ∅)
    …

    for all e ∈ G.incidentEdges(u)
      { relax edge e }
      z ← G.opposite(u,e)
      r ← getDistance(u) + weight(e)
      if r < getDistance(z)
        setDistance(z,r)
        setParent(z,e)
        Q.replaceKey(getLocator(z),r)
```

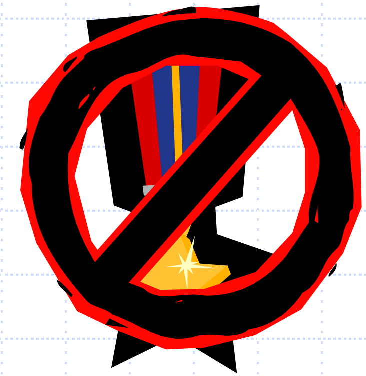
Why Dijkstra's Algorithm Works



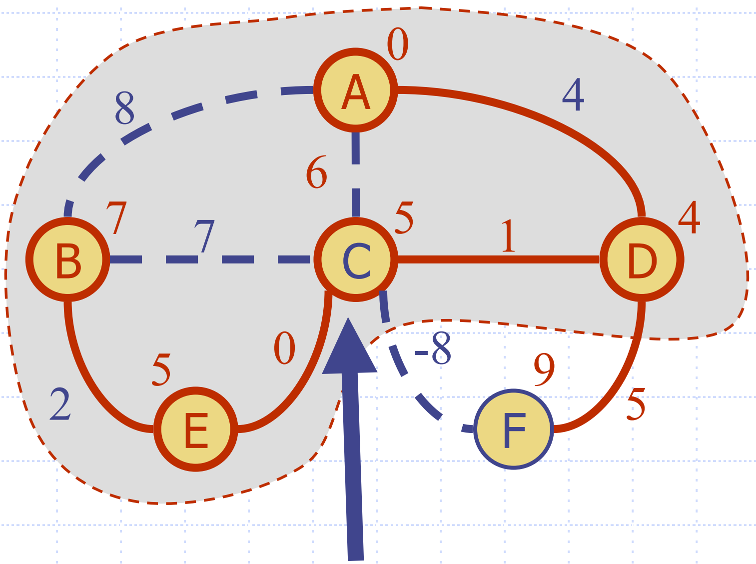
- Dijkstra's algorithm is based on the greedy method. It adds vertices by increasing distance.
  - Suppose it didn't find all shortest distances. Let F be the first wrong vertex the algorithm processed.
  - When the previous node, D, on the true shortest path was considered, its distance was correct.
  - But the edge (D,F) was **relaxed** at that time!
  - Thus, so long as d(F)≥d(D), F's distance cannot be wrong. That is, there is no wrong vertex.



Why It Doesn't Work for Negative-Weight Edges

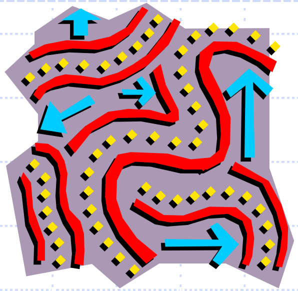


- Dijkstra's algorithm is based on the greedy method. It adds vertices by increasing distance.
  - If a node with a negative incident edge were to be added late to the cloud, it could mess up distances for vertices already in the cloud.



C's true distance is 1, but it is already in the cloud with d(C)=5!

Bellman-Ford Algorithm

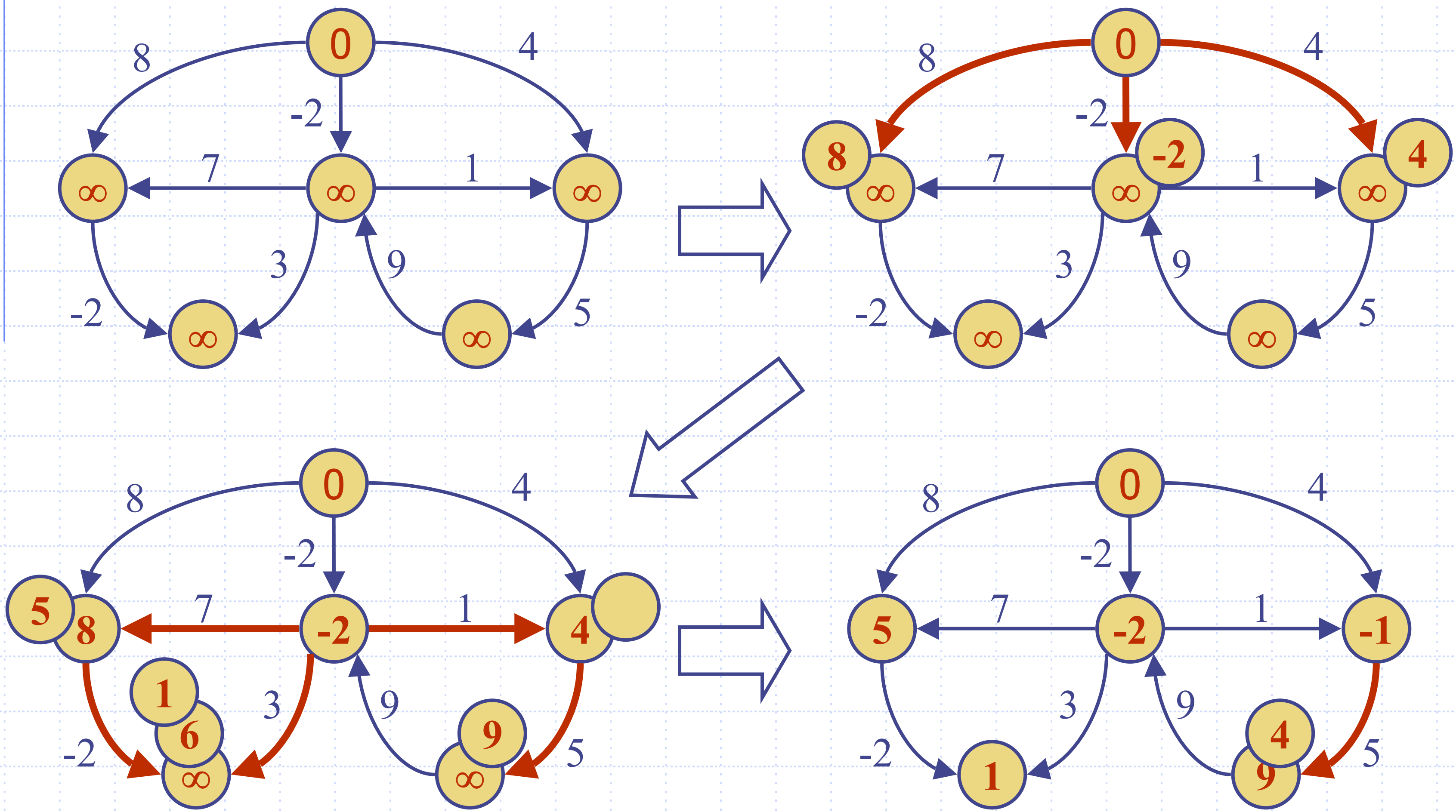


- Works even with negative-weight edges
- Must assume directed edges (for otherwise we would have negative-weight cycles)
- Iteration i finds all shortest paths that use i edges.
- Running time: O(nm).
- Can be extended to detect a negative-weight cycle if it exists
  - How?

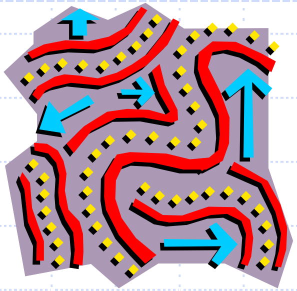
```
Algorithm BellmanFord(G, s)
  for all v ∈ G.vertices()
    if v = s
      setDistance(v, 0)
    else
      setDistance(v, ∞)
  for i ← 1 to n-1 do
    for each e ∈ G.edges()
      { relax edge e }
      u ← G.origin(e)
      z ← G.opposite(u,e)
      r ← getDistance(u) + weight(e)
      if r < getDistance(z)
        setDistance(z,r)
```

Bellman-Ford Example

Nodes are labeled with their d(v) values



DAG-based Algorithm

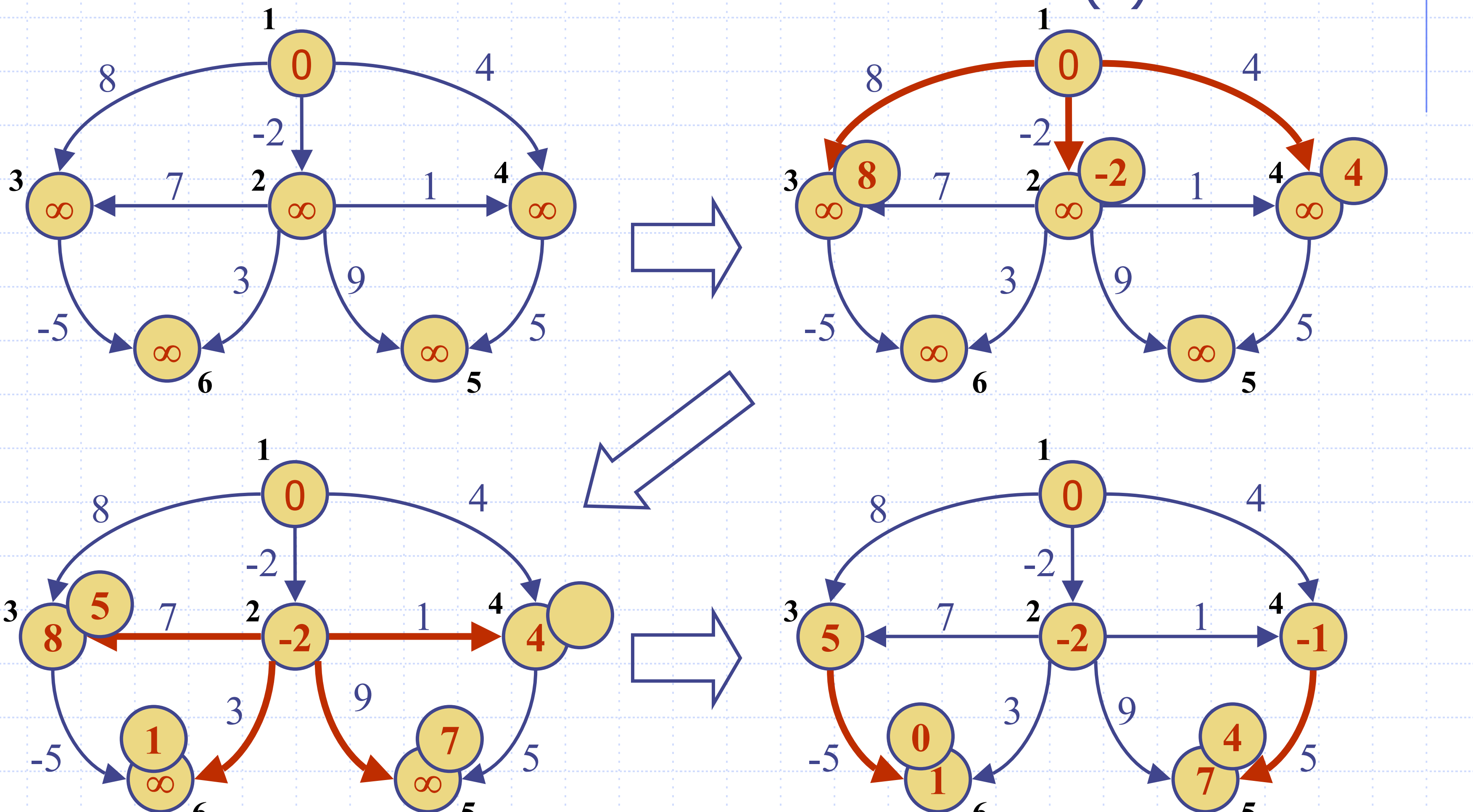


- Works even with negative-weight edges
- Uses topological order
- Doesn't use any fancy data structures
- Is much faster than Dijkstra's algorithm
- Running time: O(n+m).

```
Algorithm DagDistances(G, s)
   for all  v ∈ G.vertices()
      if  v = s
         setDistance(v, 0)
      else
         setDistance(v, ∞)
   Perform a topological sort of the vertices
   for u ← 1 to n do     {in topological order}
      for each  e ∈ G.outEdges(u)
         { relax edge e }
         z ← G.opposite(u,e)
         r ← getDistance(u) + weight(e)
         if  r < getDistance(z)
            setDistance(z,r)
```

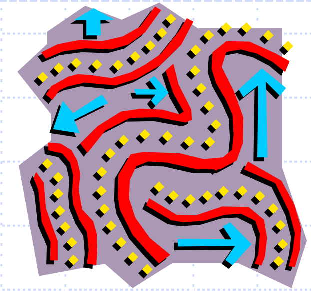
DAG Example

Nodes are labeled with their d(v) values



(two steps)

All-Pairs Shortest Paths



- Find the distance between every pair of vertices in a weighted directed graph G.
- We can make n calls to Dijkstra's algorithm (if no negative edges), which takes O(nmlog n) time.
- Likewise, n calls to Bellman-Ford would take $O(n^2m)$ time.
- We can achieve $O(n^3)$ time using dynamic programming (similar to the Floyd-Warshall algorithm).

```
Algorithm AllPair(G) {assumes vertices 1,…,n}
 for all vertex pairs (i,j)
   if i = j
      D_0[i,i] ← 0
   else if (i,j) is an edge in G
      D_0[i,j] ← weight of edge (i,j)
   else
      D_0[i,j] ← + ∞
 for k ← 1 to n do
   for i ← 1 to n do
     for j ← 1 to n do
       D_k[i,j] ← min{D_{k-1}[i,j], D_{k-1}[i,k]+D_{k-1}[k,j]}
 return D_n
```