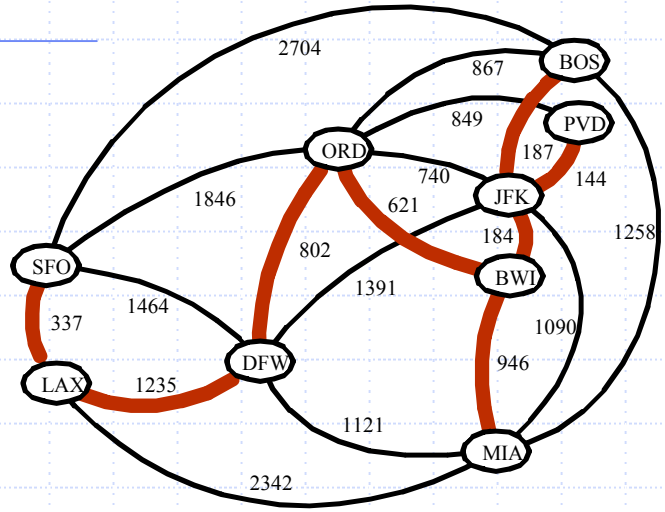
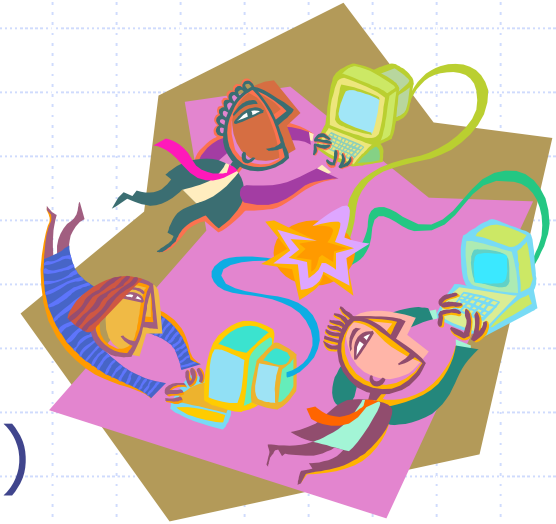
Minimum Spanning Trees



Outline and Reading



- Minimum Spanning Trees (§12.7)
  - Definitions
  - A crucial fact
- The Prim-Jarnik Algorithm (§12.7.2)
- Kruskal's Algorithm (§12.7.1)
- Baruvka's Algorithm

Minimum Spanning Tree

Spanning subgraph

- Subgraph of a graph G containing all the vertices of G

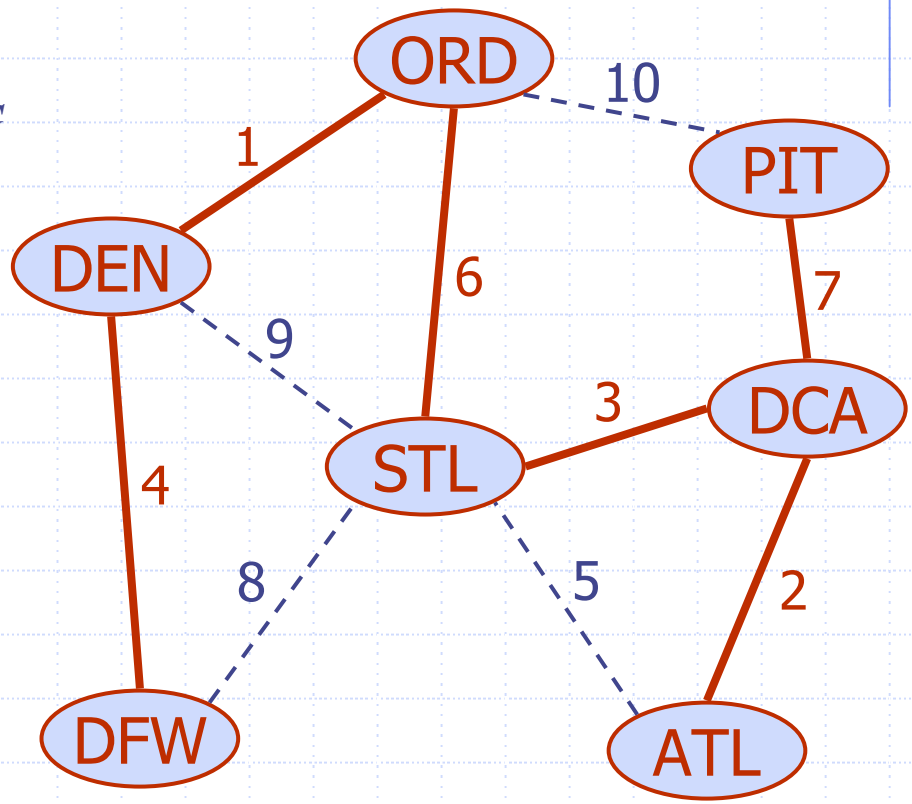
Spanning tree

- Spanning subgraph that is itself a (free) tree

Minimum spanning tree (MST)

- Spanning tree of a weighted graph with minimum total edge weight

- Applications
  - Communications networks
  - Transportation networks



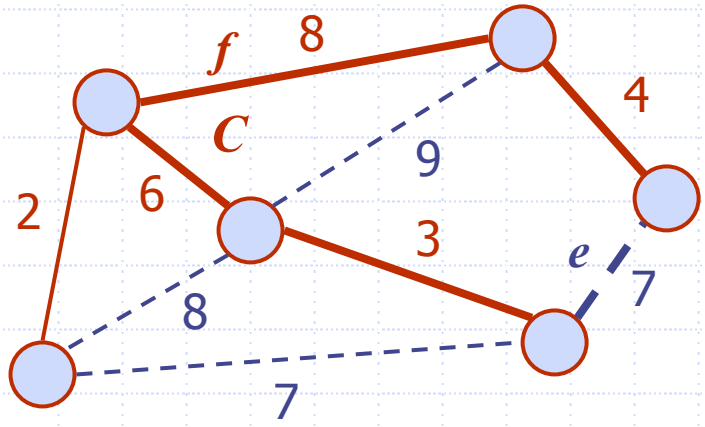
Cycle Property

Cycle Property:

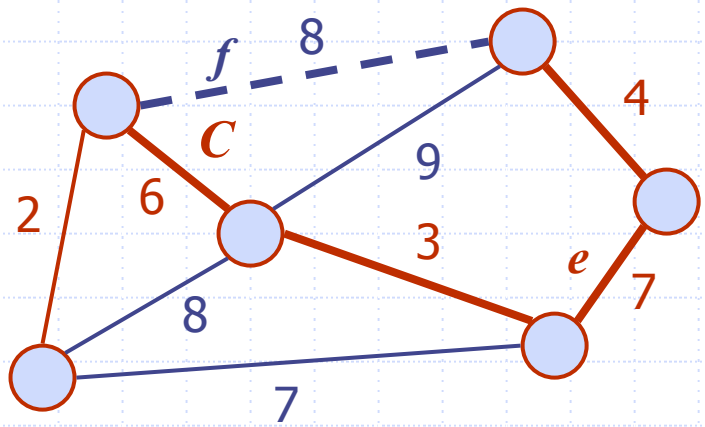
- Let T be a minimum spanning tree of a weighted graph G
- Let e be an edge of G that is not in T and C let be the cycle formed by e with T
- For every edge f of C, $weight(f) \leq weight(e)$

Proof:

- By contradiction
- If $weight(f) > weight(e)$ we can get a spanning tree of smaller weight by replacing e with f



Replacing f with e yields a better spanning tree



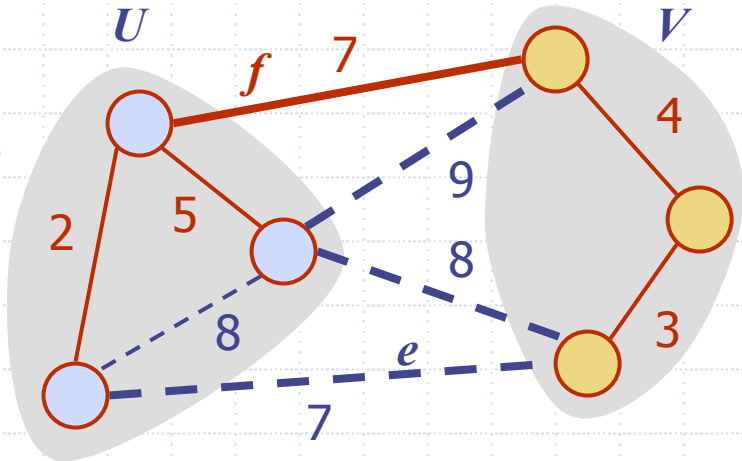
Partition Property

Partition Property:

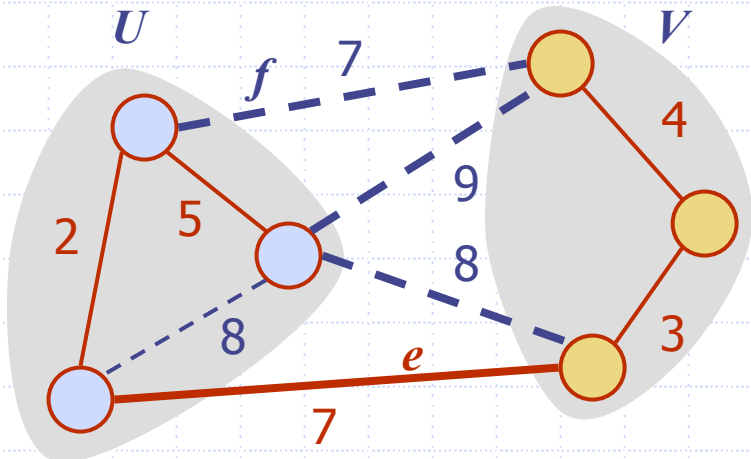
- Consider a partition of the vertices of G into subsets U and V
- Let e be an edge of minimum weight across the partition
- There is a minimum spanning tree of G containing edge e

Proof:

- Let T be an MST of G
- If T does not contain e, consider the cycle C formed by e with T and let f be an edge of C across the partition
- By the cycle property,
$$weight(f) \leq weight(e)$$
- Thus, $weight(f) = weight(e)$
- We obtain another MST by replacing f with e

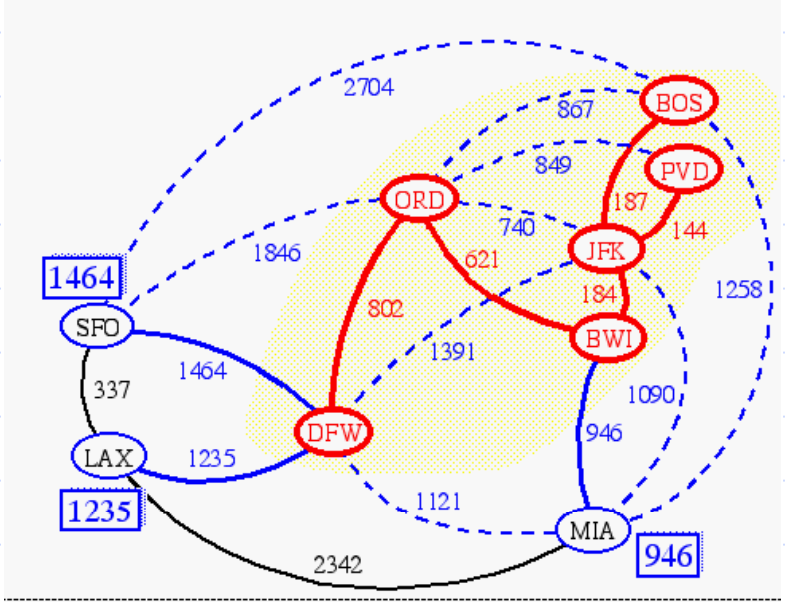


Replacing f with e yields another MST



Prim-Jarnik's Algorithm

- Similar to Dijkstra's algorithm (for a connected graph)
- We pick an arbitrary vertex s and we grow the MST as a cloud of vertices, starting from s
- We store with each vertex v a label $d(v)$ = the smallest weight of an edge connecting v to a vertex in the cloud
- At each step:
  - We add to the cloud the vertex u outside the cloud with the smallest distance label
  - We update the labels of the vertices adjacent to u

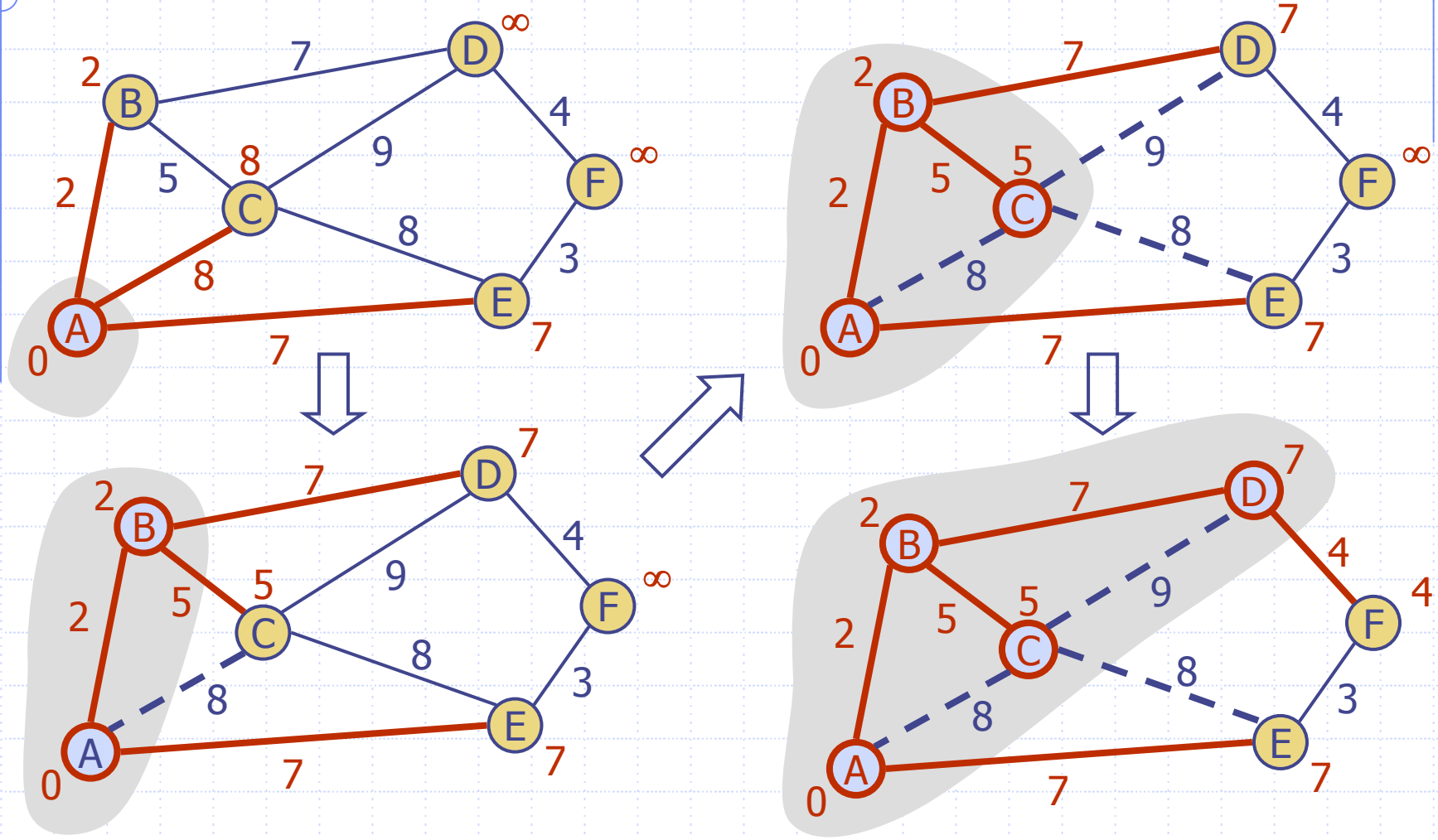


Prim-Jarnik's Algorithm (cont.)

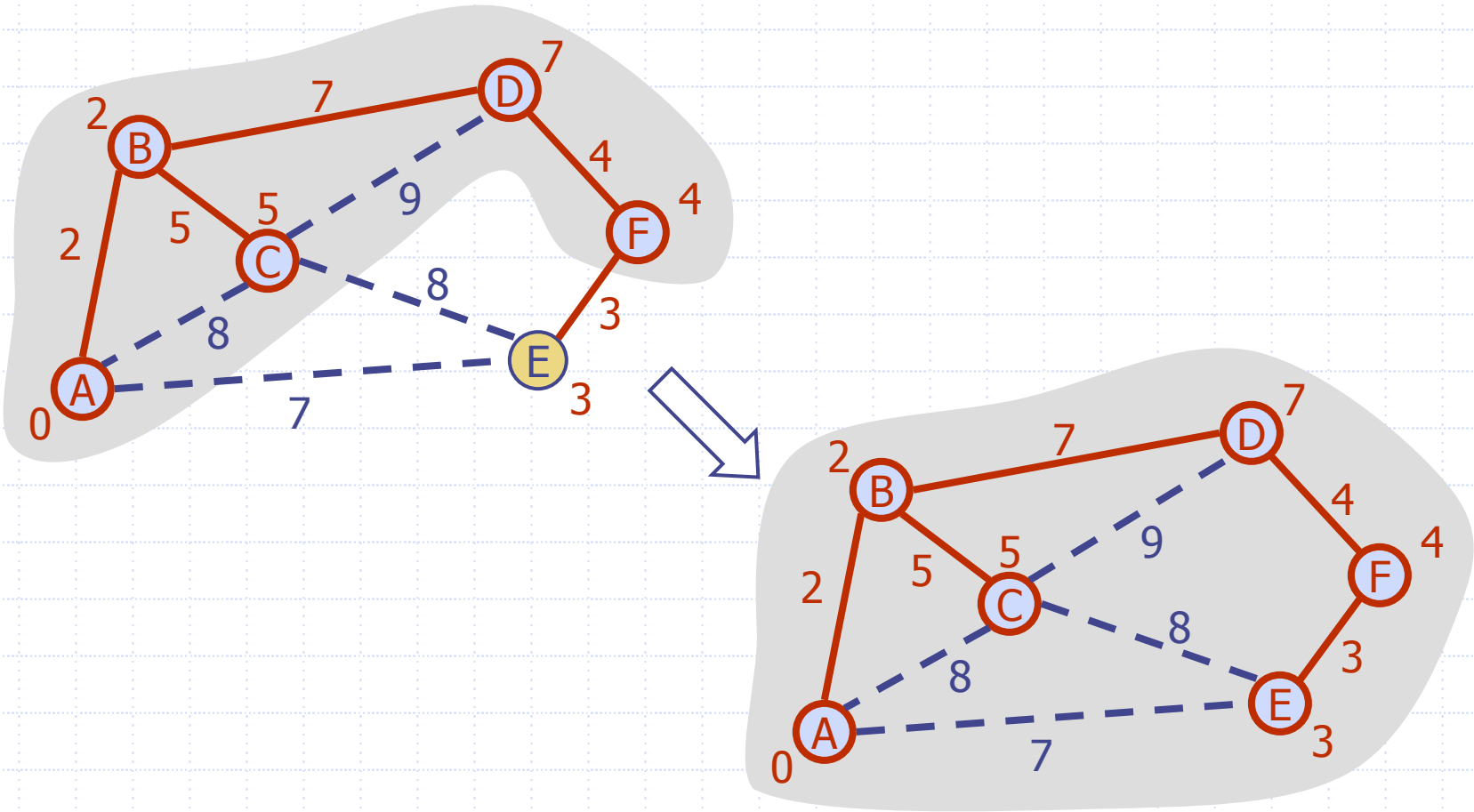
- A priority queue stores the vertices outside the cloud
  - Key: distance
  - Element: vertex
- Locator-based methods
  - *insert*(*k*,*e*) returns a locator
  - *replaceKey*(*l*,*k*) changes the key of an item
- We store three labels with each vertex:
  - Distance
  - Parent edge in MST
  - Locator in priority queue

```
Algorithm PrimJarnikMST(G)
    Q ← new heap-based priority queue
    s ← a vertex of G
    for all v ∈ G.vertices()
        if v = s
            setDistance(v, 0)
        else
            setDistance(v, ∞)
        setParent(v, ∅)
        l ← Q.insert(getDistance(v), v)
        setLocator(v,l)
    while ¬Q.isEmpty()
        u ← Q.removeMin()
        for all e ∈ G.incidentEdges(u)
            z ← G.opposite(u,e)
            r ← weight(e)
            if r < getDistance(z)
                setDistance(z,r)
                setParent(z,e)
                Q.replaceKey(getLocator(z),r)
```

Example



Example (contd.)



Analysis

- Graph operations
  - Method incidentEdges is called once for each vertex
- Label operations
  - We set/get the distance, parent and locator labels of vertex z $O(\deg(z))$ times
  - Setting/getting a label takes $O(1)$ time
- Priority queue operations
  - Each vertex is inserted once into and removed once from the priority queue, where each insertion or removal takes $O(\log n)$ time
  - The key of a vertex w in the priority queue is modified at most $\deg(w)$ times, where each key change takes $O(\log n)$ time
- Prim-Jarnik's algorithm runs in $O((n + m) \log n)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\sum_v \deg(v) = 2m$
- The running time is $O(m \log n)$ since the graph is connected

Kruskal's Algorithm

- A priority queue stores the edges outside the cloud
  - Key: weight
  - Element: edge
- At the end of the algorithm
  - We are left with one cloud that encompasses the MST
  - A tree T which is our MST

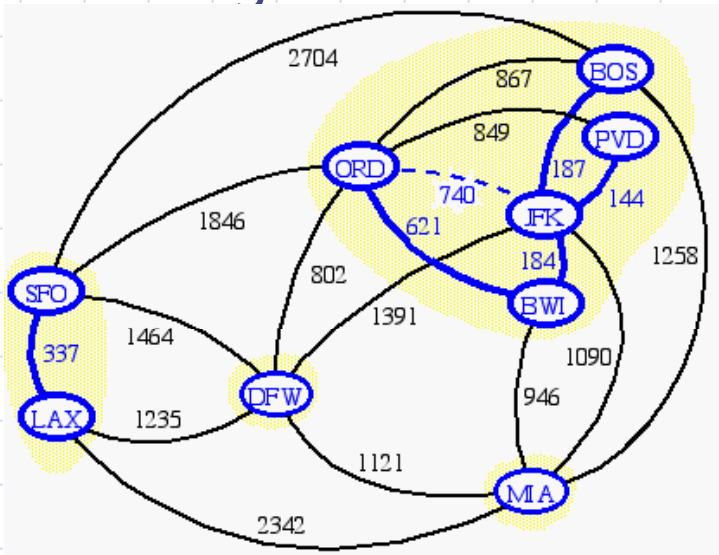
Algorithm *KruskalMST*(*G*)
  for each vertex *V* in *G* **do**
    define a *Cloud(v)* of ← {*v*}
  let *Q* be a priority queue.
  Insert all edges into *Q* using their weights as the key
  T ← ∅
  while *T* has fewer than *n*-1 edges **do**
    edge *e* = *T.removeMin()*
    Let *u*, *v* be the endpoints of *e*
    if *Cloud(v)* ≠ *Cloud(u)* **then**
      Add edge *e* to *T*
      Merge *Cloud(v)* and *Cloud(u)*
  return *T*

Data Structure for Kruskal Algortihm

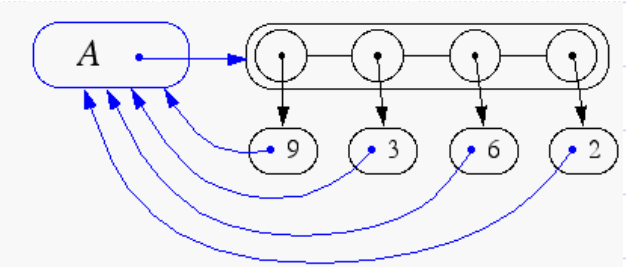
- The algorithm maintains a forest of trees
- An edge is accepted it if connects distinct trees
- We need a data structure that maintains a **partition**, i.e., a collection of disjoint sets, with the operations:

  -**find**(u): return the set storing u

  -**union**(u,v): replace the sets storing u and v with their union



Representation of a Partition



- Each set is stored in a sequence
- Each element has a reference back to the set
  - operation find(u) takes O(1) time, and returns the set of which u is a member.
  - in operation union(u,v), we move the elements of the smaller set to the sequence of the larger set and update their references
  - the time for operation union(u,v) is $\min(n_u,n_v)$, where n_u and n_v are the sizes of the sets storing u and v
- Whenever an element is processed, it goes into a set of size at least double, hence each element is processed at most log n times

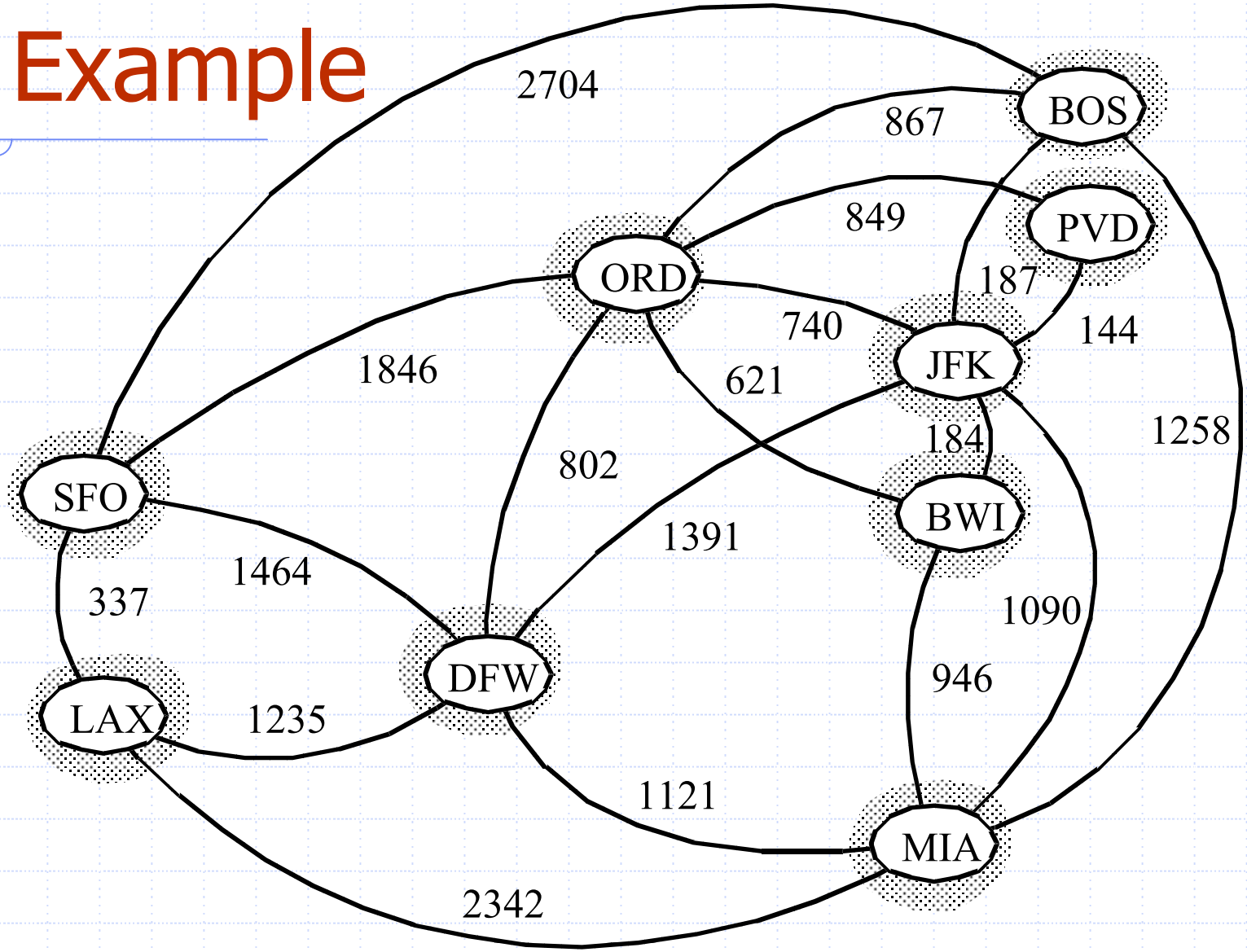
Partition-Based Implementation

- A partition-based version of Kruskal's Algorithm performs cloud merges as unions and tests as finds.

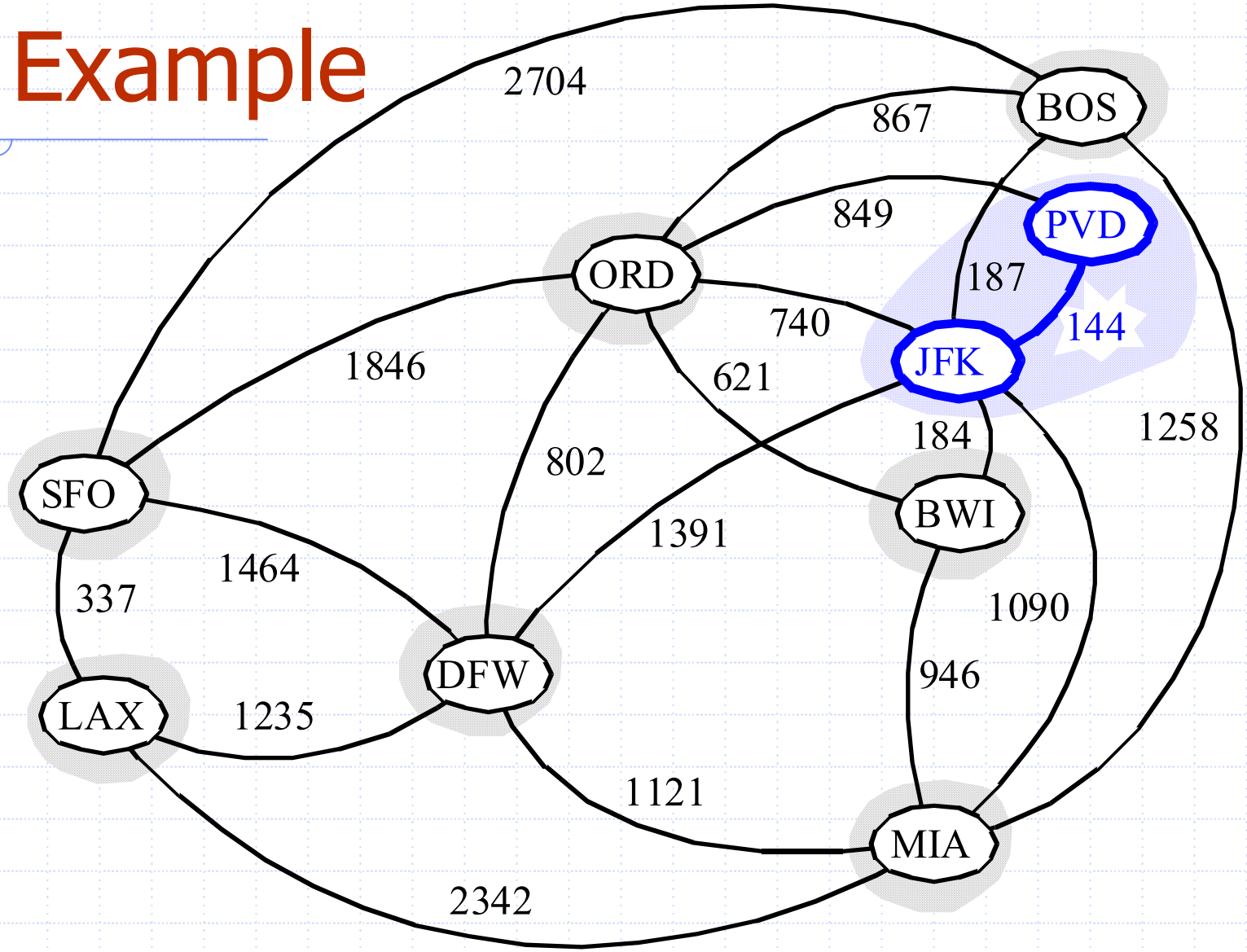
Algorithm Kruskal(*G*):
  Input: A weighted graph *G*.
  Output: An MST *T* for *G*.
Let *P* be a partition of the vertices of *G*, where each vertex forms a separate set.
Let *Q* be a priority queue storing the edges of *G*, sorted by their weights
Let *T* be an initially-empty tree
while *Q* is not empty **do**
  (*u*,*v*) ← *Q*.removeMinElement()
  if *P*.find(*u*) != *P*.find(*v*) **then**
    Add (*u*,*v*) to *T*
    P.union(*u*,*v*)
return *T*

Running time:
O((n+m)log n)

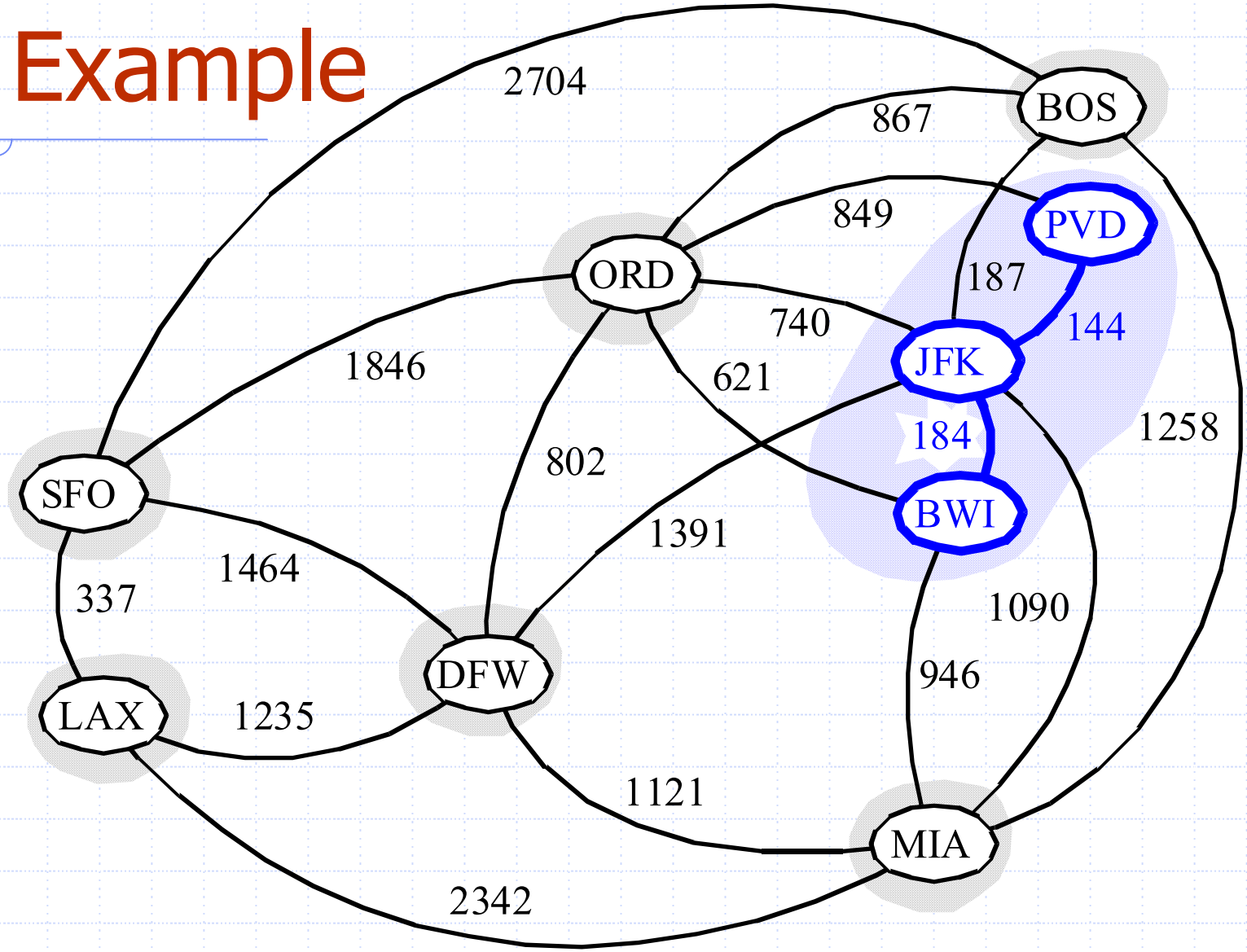
Kruskal Example



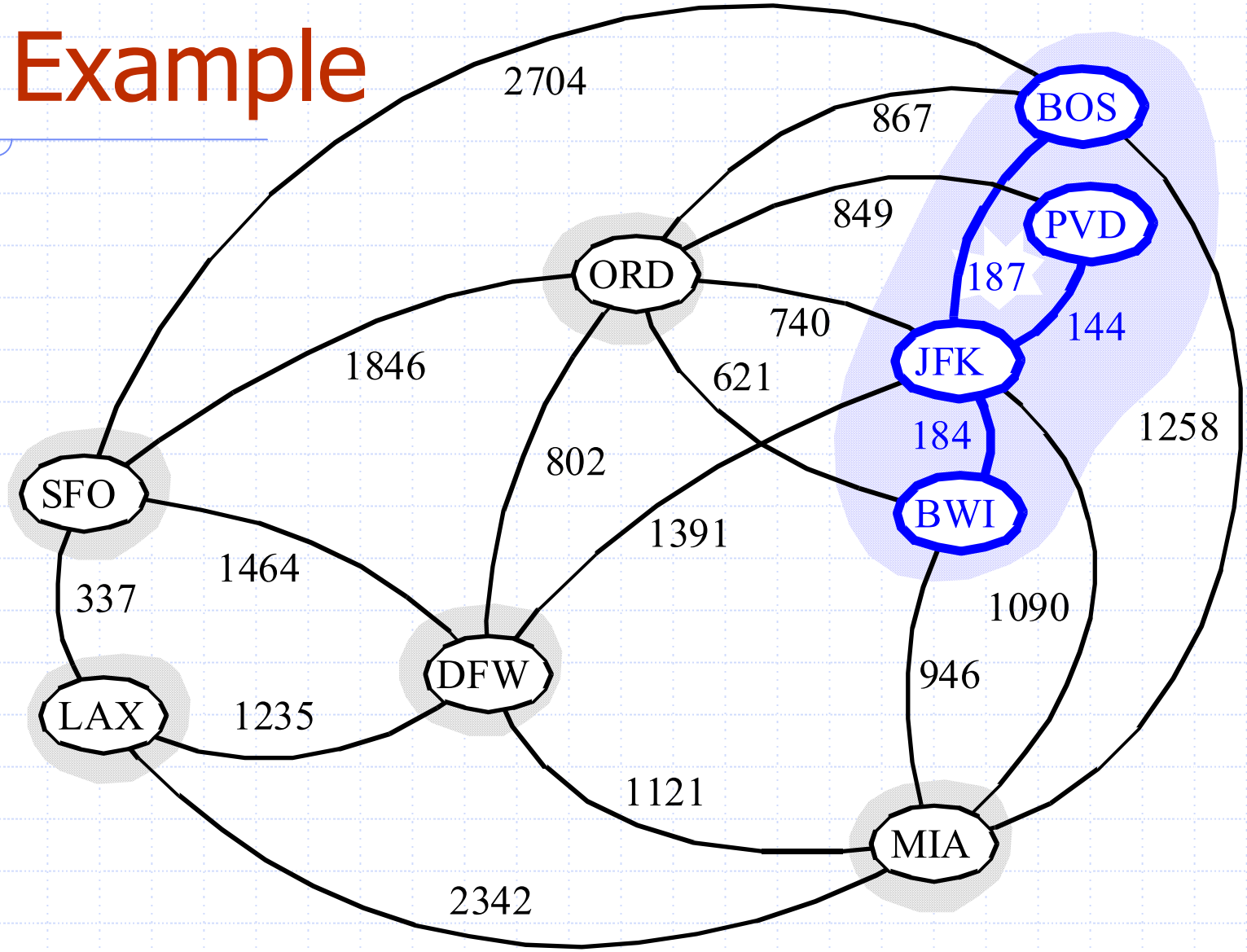
Example


2704
867
BOS
849
PVD
ORD
187
740
144
JFK
1846
621
184
1258
802
SFO
BWI
1391
1464
337
1090
DFW
946
LAX
1235
1121
MIA
2342

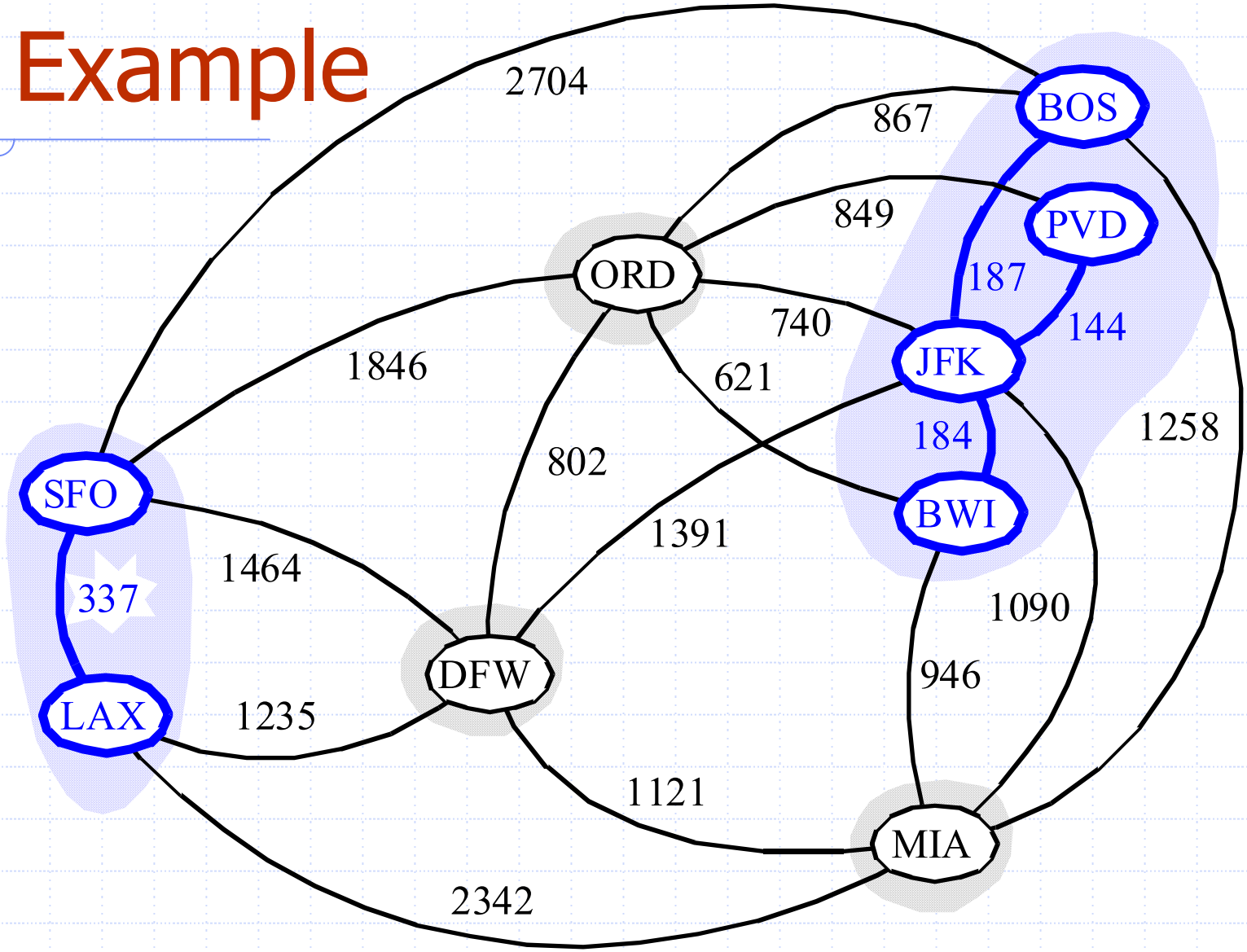
Example


2704
867
BOS
849
PVD
ORD
187
740
144
JFK
1846
621
1258
184
802
SFO
BWI
1391
1464
337
1090
DFW
946
LAX
1235
1121
MIA
2342

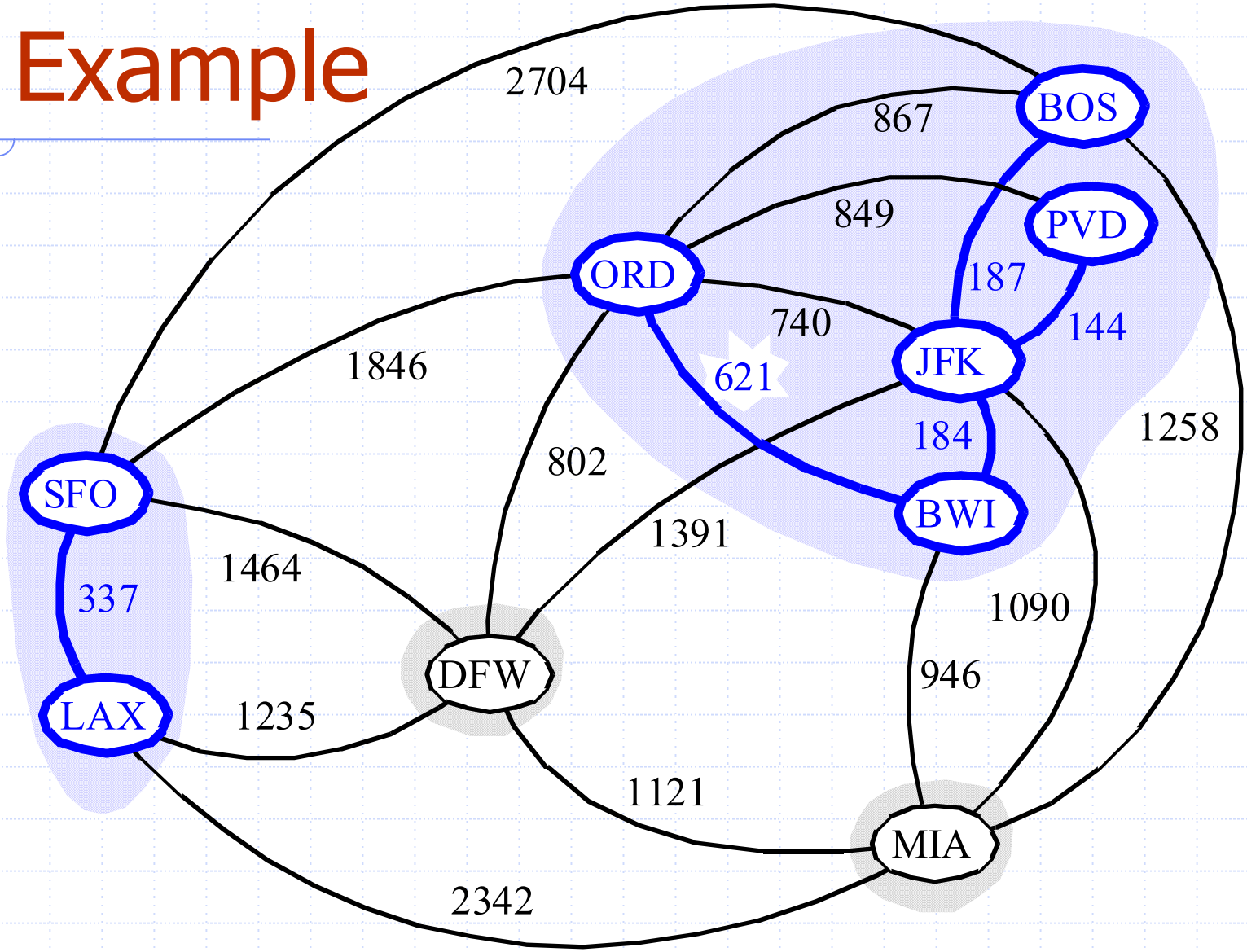
Example


2704
867
849
187
144
740
1846
621
1258
184
802
1391
1464
337
1090
946
1235
1121
2342
BOS
PVD
ORD
JFK
SFO
BWI
DFW
LAX
MIA

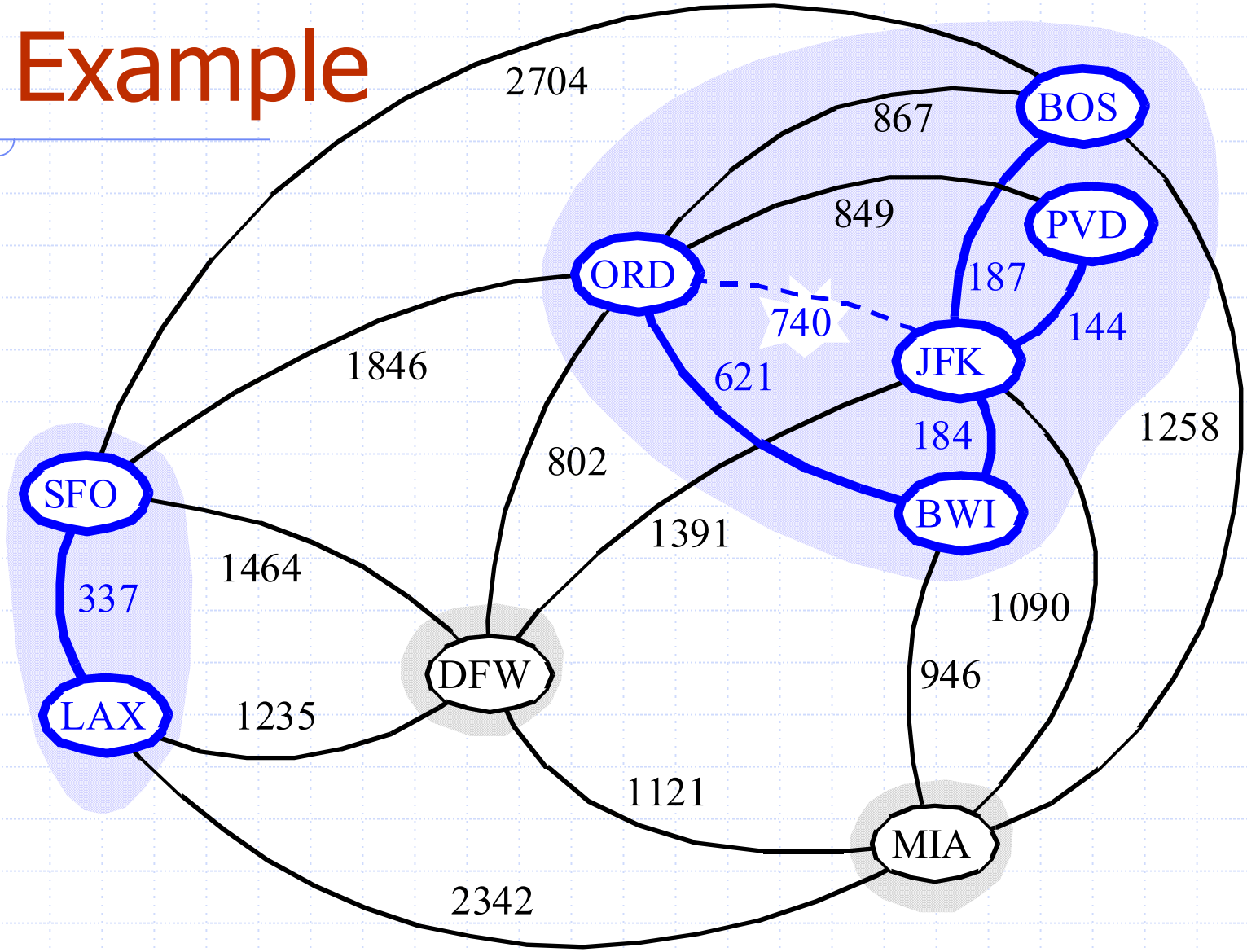
Example


2704
867
BOS
849
PVD
ORD
187
144
740
JFK
1846
621
184
1258
802
SFO
BWI
1391
1464
337
1090
DFW
946
LAX
1235
1121
MIA
2342

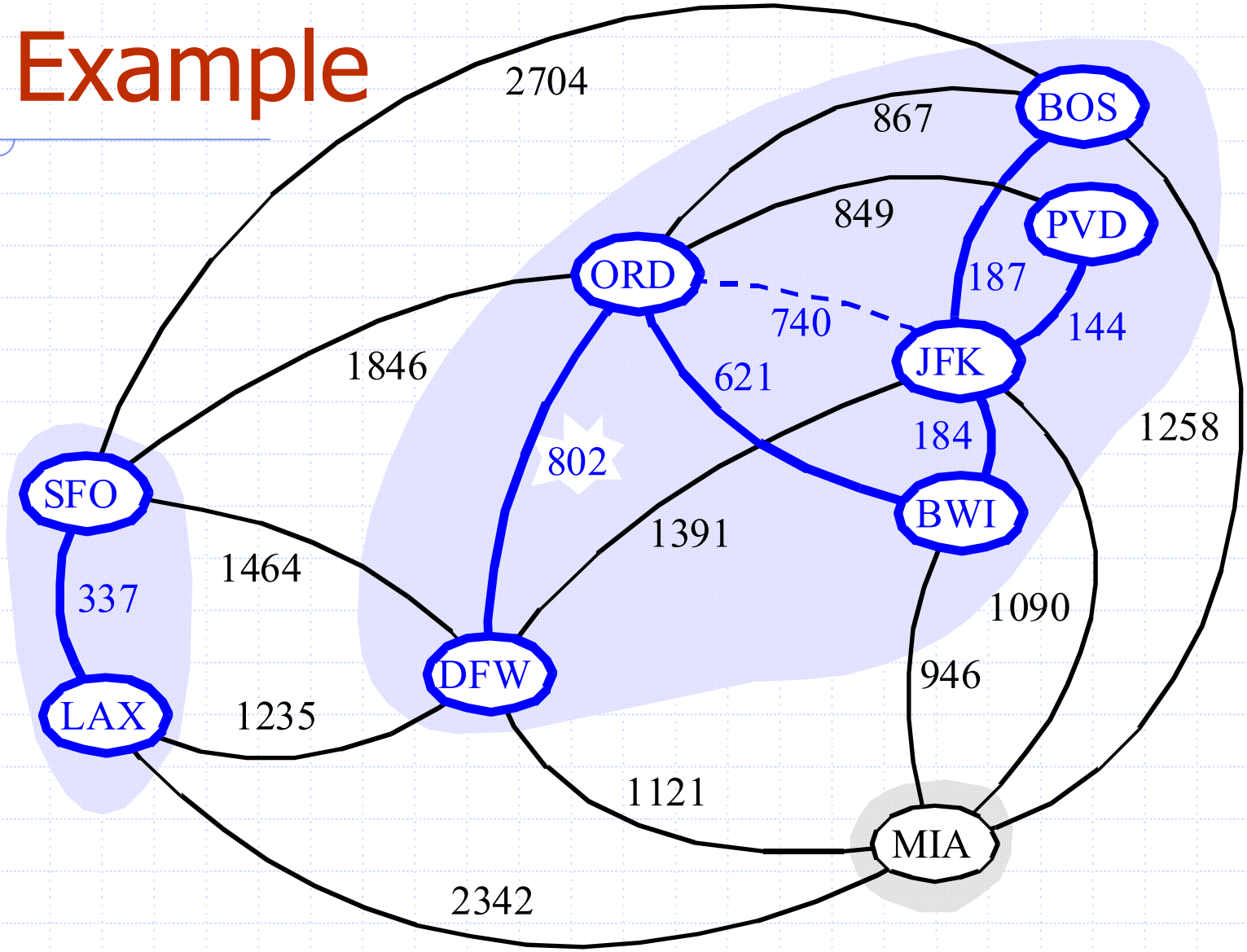
Example


2704
867
BOS
849
PVD
ORD
187
740
144
1846
621
JFK
184
1258
802
SFO
BWI
1391
1464
337
1090
DFW
946
LAX
1235
1121
MIA
2342

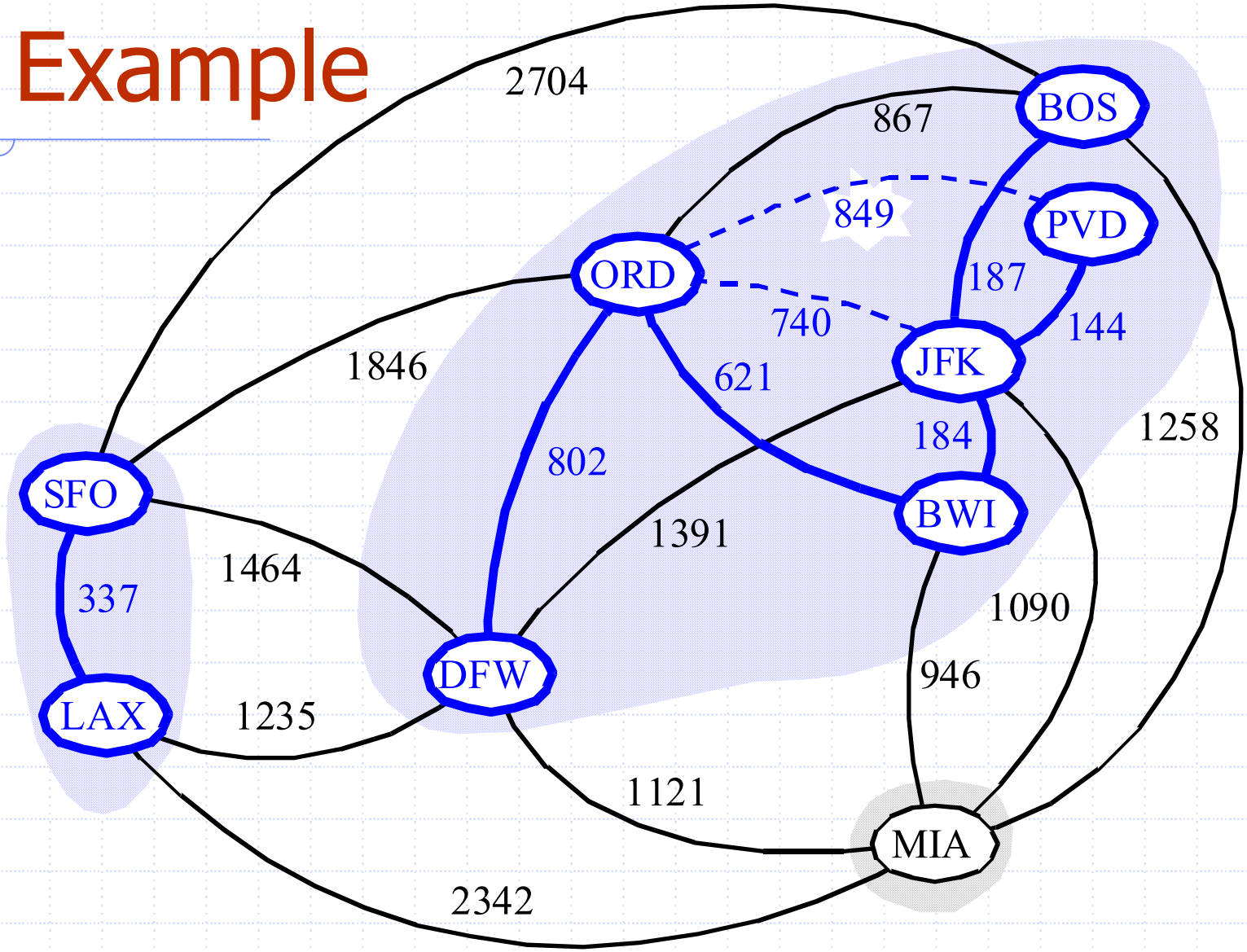
Example


2704
867
BOS
849
PVD
ORD
187
740
144
JFK
1846
621
184
1258
802
SFO
BWI
1391
1464
337
1090
DFW
946
LAX
1235
1121
MIA
2342

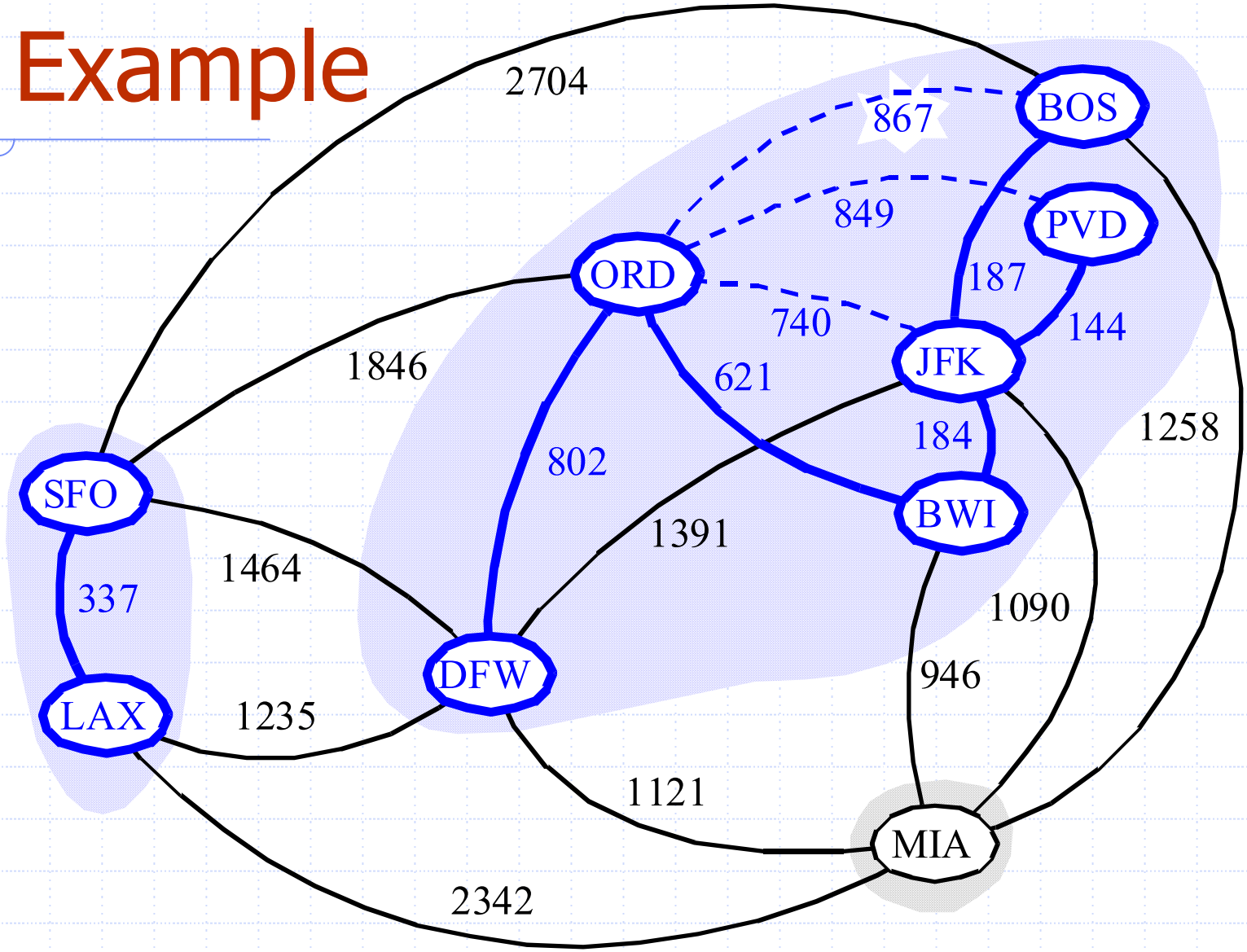
Example


2704
867
BOS
849
PVD
ORD
187
740
144
JFK
1846
621
184
1258
802
SFO
BWI
1391
1464
337
1090
DFW
946
LAX
1235
1121
MIA
2342

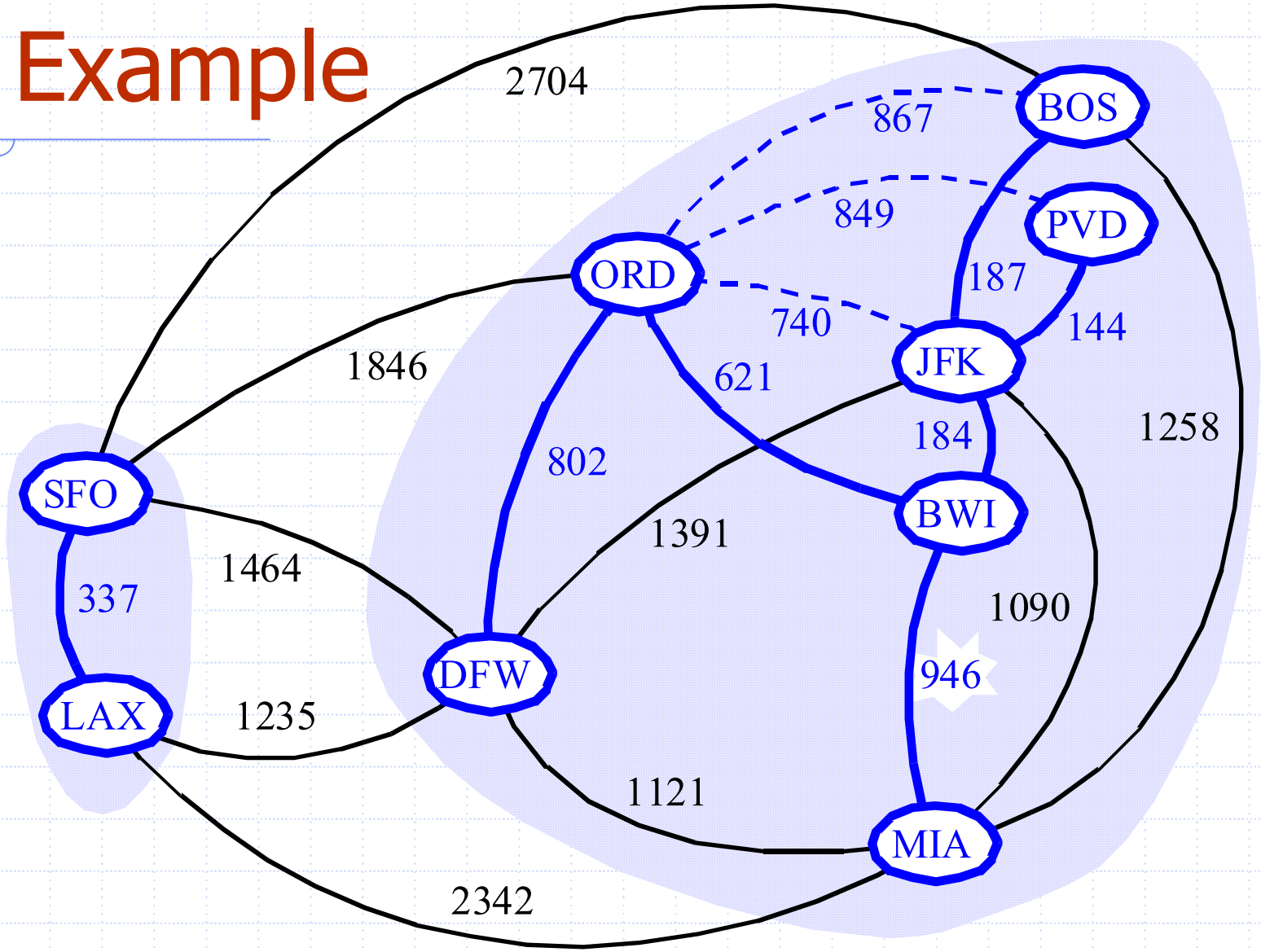
Example


2704
867
BOS
849
PVD
ORD
187
740
144
JFK
1846
621
184
1258
802
SFO
BWI
1391
1464
337
1090
DFW
946
LAX
1235
1121
MIA
2342

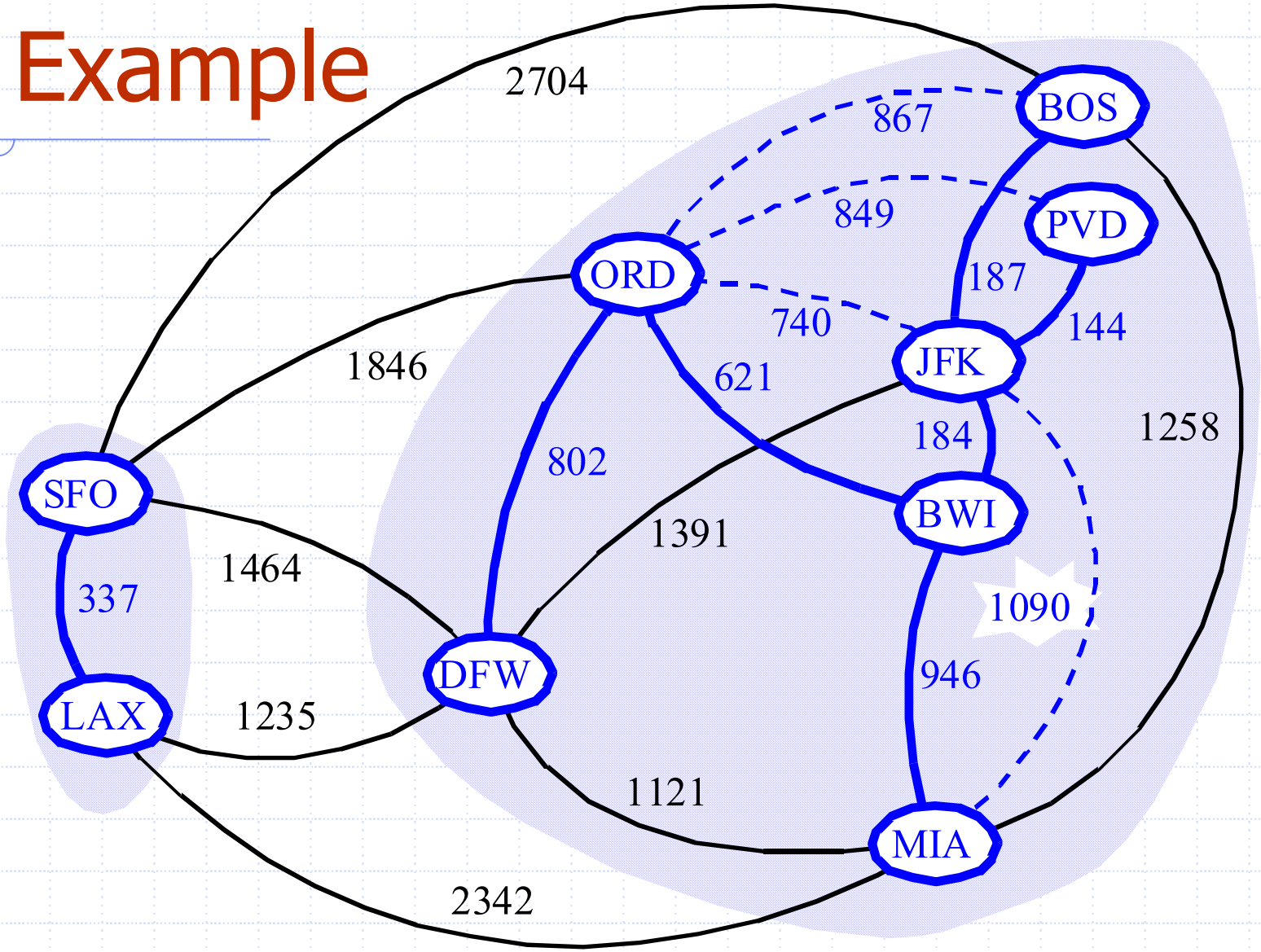
Example


2704
867
BOS
849
PVD
ORD
187
740
144
JFK
1846
621
184
1258
802
SFO
BWI
1391
1464
337
1090
DFW
946
LAX
1235
1121
MIA
2342

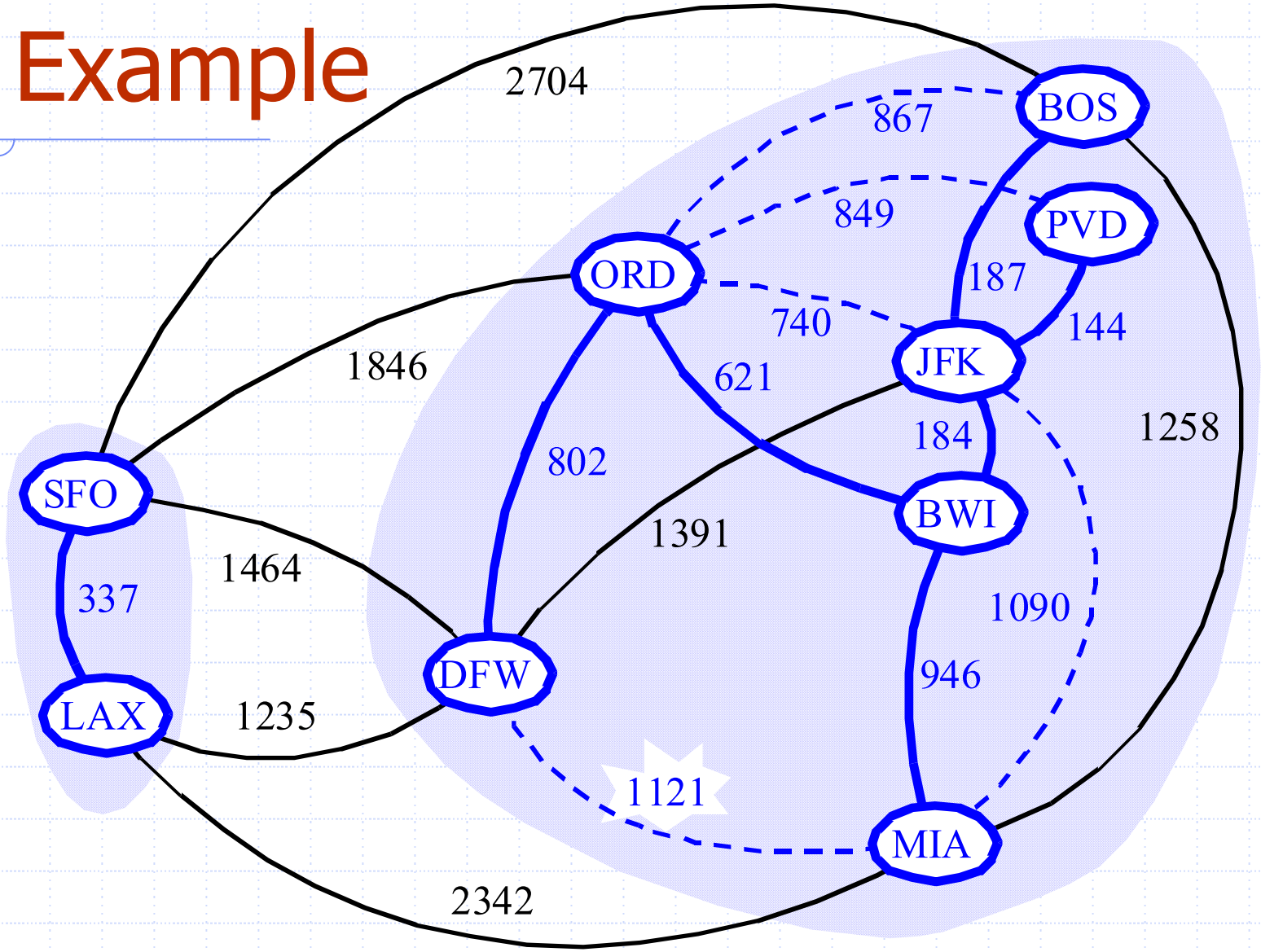
Example


2704
867
BOS
849
PVD
ORD
187
740
144
JFK
1846
621
184
1258
802
SFO
BWI
1391
1464
337
1090
946
DFW
LAX
1235
1121
MIA
2342

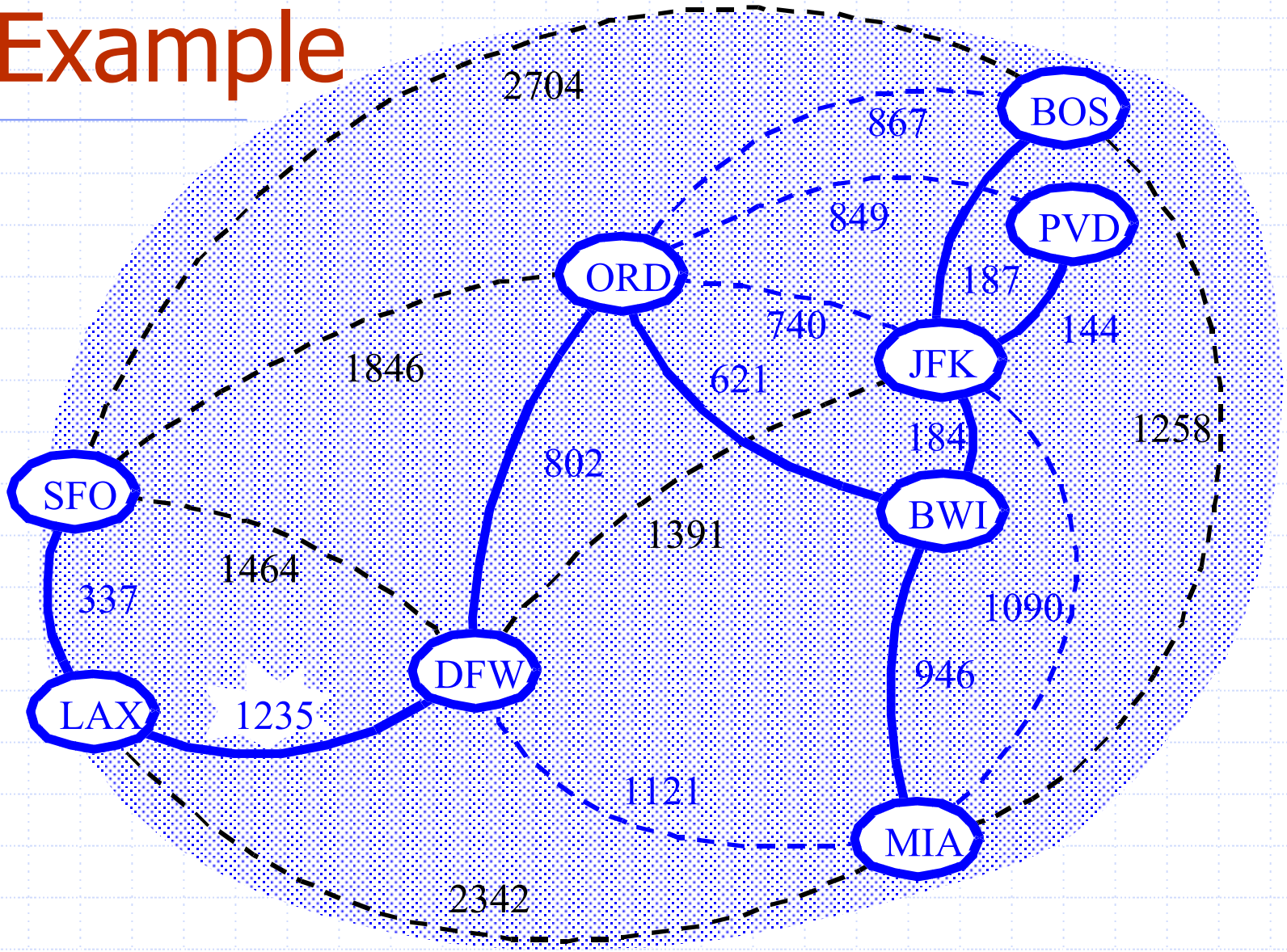
Example


2704
867
BOS
849
PVD
ORD
187
740
144
JFK
1846
621
1258
184
802
SFO
BWI
1391
1464
337
1090
946
DFW
LAX
1235
1121
MIA
2342

Example


2704
867
BOS
849
PVD
ORD
187
740
144
JFK
1846
621
1258
184
802
SFO
BWI
1391
1464
337
1090
DFW
946
LAX
1235
1121
MIA
2342

Example



2704
BOS
867
849
PVD
ORD
187
740
144
JFK
1846
621
1258
184
802
SFO
BWI
1391
1464
337
1090
DFW
946
LAX
1235
1121
MIA
2342

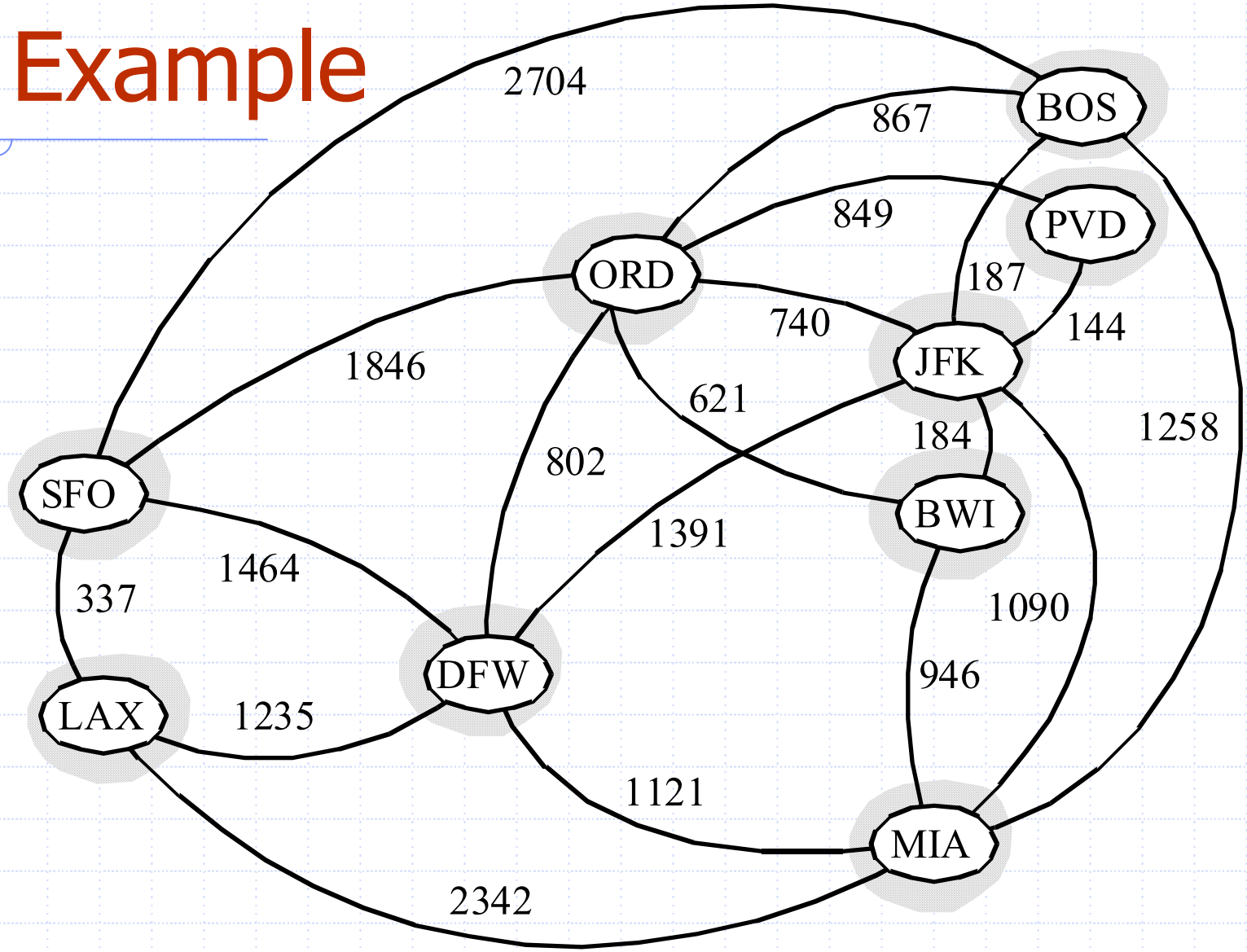
Baruvka's Algorithm

- Like Kruskal's Algorithm, Baruvka's algorithm grows many "clouds" at once.

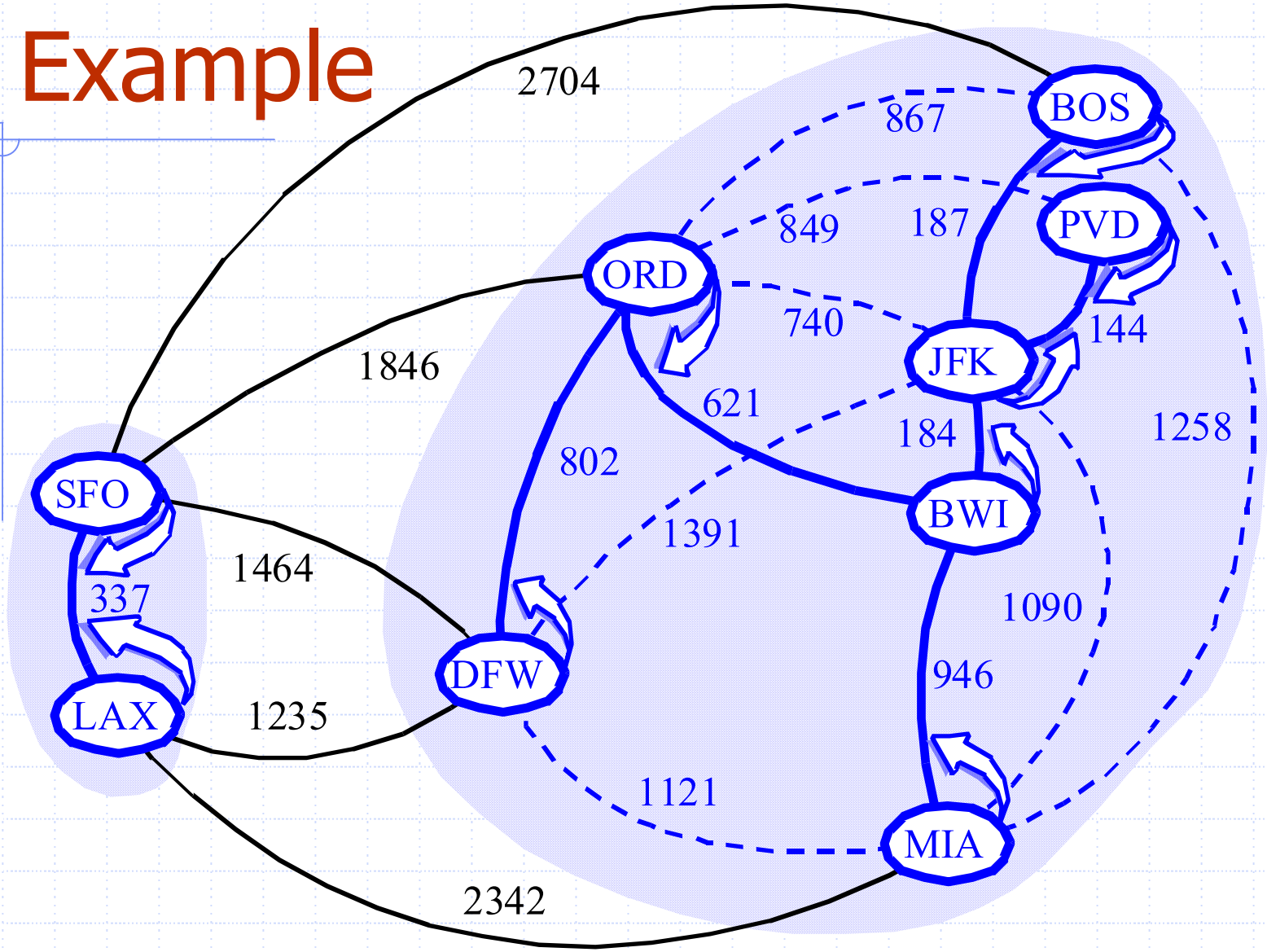
```
Algorithm BaruvkaMST(G)
    T ← V  {just the vertices of G}
  while T has fewer than n-1 edges do
    for each connected component C in T do
      Let edge e be the smallest-weight edge from C to another component in T.
      if e is not already in T then
        Add edge e to T
  return T
```

- Each iteration of the while-loop halves the number of connected compontents in T.
  - The running time is O(m log n).

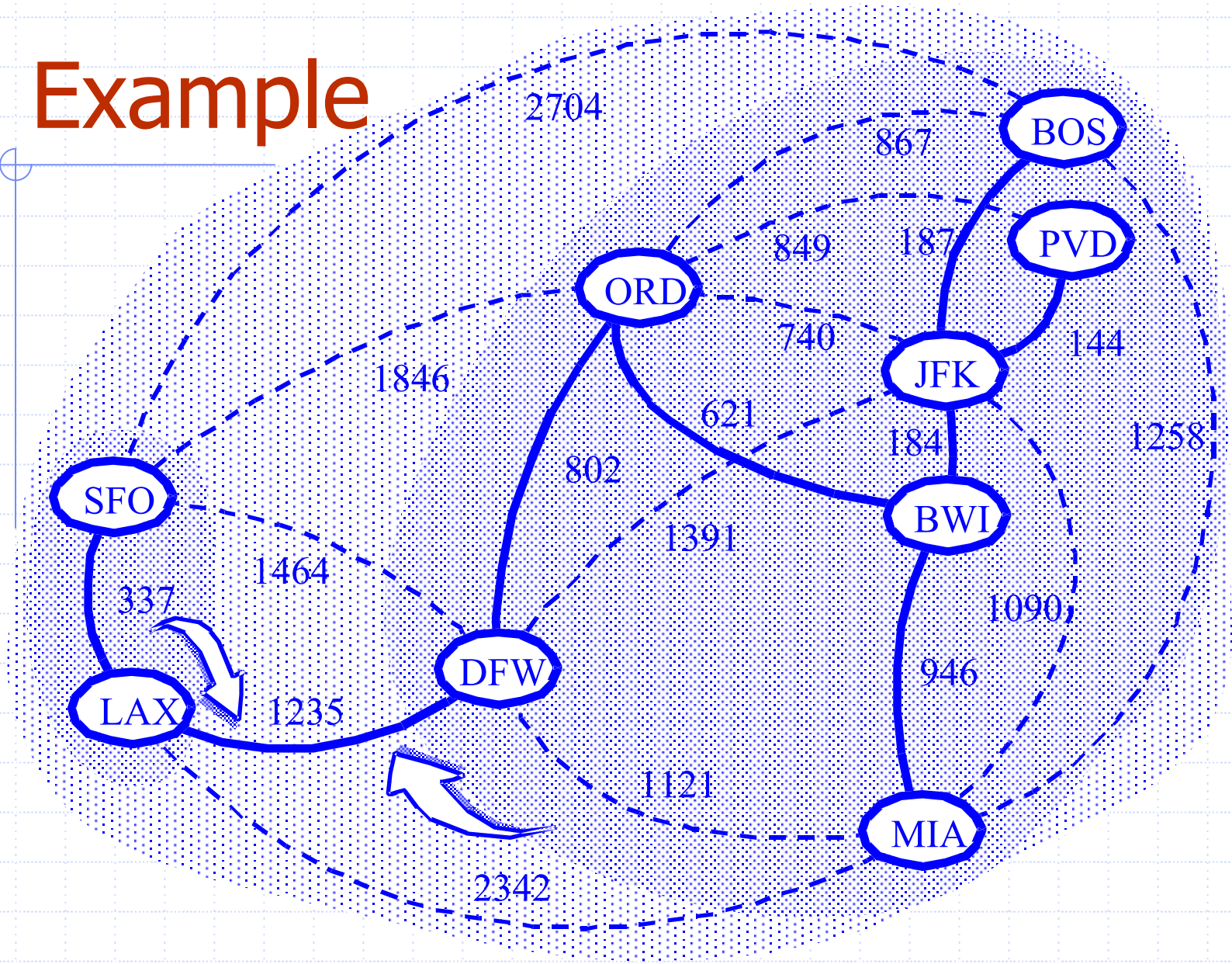
Baruvka Example



Example


2704
867
BOS
849
187
PVD
ORD
740
144
JFK
1846
621
184
1258
802
SFO
BWI
1391
1464
337
1090
DFW
946
LAX
1235
1121
MIA
2342

Example


2704
867
BOS
849
187
PVD
ORD
740
144
1846
JFK
621
184
1258
802
SFO
BWI
1391
1464
337
1090
DFW
946
LAX
1235
1121
MIA
2342