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Skiba points for optimal investment strategies

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1 Introduction

If at a given point in time, a firm's capital stock is not optimal, that is, if the marginal product of capital does not equal the marginal cost of capital, a manager has to take a decision about investment or disinvestment. Under the assumption that there adjustment costs to be paid on changing the capital stock, it is not optimal to try to jump to the optimal level all at once; rather, the manager has to make an investment plan. Increasing returns to scale over some regions of the firm size may lead to the existence of several optimal long term steady states. The initial size of the firm determining to which of these states an optimal investment plan will lead. This divides the possible initial sizes into "optimal basins of attraction" of the respective long term steady states. These basins are separated by threshold points, which are often called Skiba or Dechert-Nishimura-Skiba (DNS) points, after the articles of Skiba [4] and Dechert and Nishimura [2].

In this paper we numerically compute DNS-points. All computations are done by Maple.

2 Control problem

The manager of the firm faces the optimality problem for maximizing the discounted profit functional $J(k, u)$:

$$\max_u J(k, u) = \max_u \int_0^\infty e^{-\rho t}[r(k) - c(u)]dt, \quad \rho > 0, \tag{1}$$

subject to the dynamics of capital stock:

$$\dot{k} = u - \mu k, \quad k(0) = k_0, \tag{2}$$

where $k(t) \geq 0$ denotes the capital stock and $u(t) \geq 0$ is investment. The revenue function is given by $r(k)$ and the investment costs are $c(u)$, where

$$\begin{aligned} &r(k) = \max\{r_1(k), r_2(k)\}, r_1(k) = a_1k^2 + b_1k,\ r_2(k) = a_2k^2 + b_2k + c_2, \\ &c(u) = a_3u^2 + b_3u. \end{aligned} \tag{3}$$

The coefficients a_1 and a_2 are negative and a_3 is positive, i.e. $r(k)$ is a piecewise concave function while $c(u)$ is convex.

Let (k^*, r^*) be the intersection point of the parabols $r_1(k)$ and $r_2(k)$ and suppose that $r_1(k) > r_2(k)$ for $k < k^*$ and $r_1(k) < r_2(k)$ for $k > k^*$.

To determine the necessary optimal conditions we write down the Hamiltonian:

$$H(k, u, q) = r_i(k) - c(u) + q(u - \mu k), \tag{4}$$

where $i = 1$ for $k < k^*$ and $i = 2$ for $k > k^*$. For the optimal values of u and q we have

$$\frac{\partial H}{\partial u} = -c'(u) + q = 0,$$

i.e. $q = c'(u) = 2a_3 + ub_3$ and $\dot{q} = c''(u)\dot{u} = 2a_3\dot{u}$.

Also $\dot{q} = \rho q - \frac{\partial H}{\partial k} = \rho q - r_i'(k) + \mu q = (\rho + \mu)c'(u) - r_i(k) = 0$ and

$$\dot{u} = \frac{1}{2a_3}[(\rho + \mu)c'(u) - r_i'(k)]. \tag{5}$$

We can compute the optimal values Π using the Hamiltonian, since $\Pi = H/\rho$ (see [1], Lemma 3.2), i.e.

$$\rho\Pi(k, u) = H(k, u, c'(u)) = r_i(k) - c(u) - c'(u)(u - \mu k) \tag{6}$$

for some optimal u, where $i = 1$ for $k < k^*$ and $i = 2$ for $k > k^*$.

3 Computational procedure

From (2) and (5) we construct two ODE's systems:

$$\left|\begin{array}{l} \dot{k} = u - \mu k, \\ \dot{u} = \dfrac{1}{2a_3}[(\rho+\mu)(2a_3 u + b_3) - 2a_i k - b_i] \end{array}\right. \tag{7}$$

for $i = 1, 2$ which give us optimal trajectories under suitable initial conditions $k(0)$ and $u(0)$. First we find stationary points $(\bar{k}_i, \bar{u}_i), i = 1, 2$ (equilibria, steady states) of (7) from

$$\dot{k} = 0 \quad \text{and} \quad \dot{u} = 0. \tag{8}$$

It is clear that $\bar{k}_1 < k^* < \bar{k}_2$.

Now we can calculate such u_i^* that the solutions $k(t)$ of (7) for initial conditions $k(0) = k^*$ and $u(0) = u_i^*$ satisfy $k(\infty) = \lim_{t\to\infty} k(t) = \bar{k}_i$. The values u_i^* can be obtained varying $u(0)$ (using the bisection method) and repeatedly solving (7) numerically. This method works because for some small δ the solution $k(t)$ of (7) with $k(0) = k^*$ and

$$\begin{array}{llll} u(0) = u_i^* + \delta & \text{satisfies} & k(\infty) = \infty & \text{and} \\ u(0) = u_i^* - \delta & \text{satisfies} & k(\infty) = -\infty. & \end{array} \tag{9}$$

In practice for a small $\epsilon > 0$ we check that for $u(0) = u_i$ the trajectory $k(t)$ satisfies $|k(T) - \bar{k}_i| < \epsilon$ for sufficiently large $T > 0$ (see Fig. 2).

Next we establish the optimal control $\hat{u}_2$ for initial capital stocks $\bar{k}_1$ using (7) for $i = 2$ and the optimal control $\hat{u}_1$ for initial capital stocks $\bar{k}_2$ for $i = 1$.

Let us denote by Π_1 the optimal value for $r(k) = r_1(k)$ and for all $k \geq 0$ and by Π_2 the optimal value for $r(k) = r_2(k)$, also for all $k \geq 0$. We compare the values of Π_1 and Π_2 at the points $\bar{k}_1$, k^* and $\bar{k}_2$. In the case when

$$\left|\begin{array}{l} \Pi_1(\bar{k}_1, \bar{u}_1) > \Pi_2(\bar{k}_1, \hat{u}_2), \\[1ex] \Pi_1(k^*, u_1^*) < \Pi_2(k^*, u_2^*) \quad \text{and} \\[1ex] \Pi_1(\bar{k}_2, \hat{u}_1) < \Pi_2(\bar{k}_2, \bar{u}_2) \end{array}\right. \tag{10}$$

the DNS-point k_{DNS} exists, $\bar{k}_1 < k_{DNS} < k^* < \bar{k}_2$ and we can calculate it.

To find $\pi_1(t) = \Pi_1(k(t), u(t))$ we should calculate the optimal trajectory $(k(t), u_1(t))$ solving the ODE's:

$$\left|\begin{array}{l} \dot{k}(t) = -(u(t) - \mu k(t)) \\ \dot{u}(t) = -\dfrac{1}{2a_3}[(\rho+\mu)(2a_3 u(t) + b_3) - 2a_1 k(t) - b_1] \end{array}\right. \tag{11}$$

with $k(0) = k^*$ and $u(0) = u_1^*$.

To find $\pi_2(t) = \Pi_2(k(t), u_2(t))$ a new optimal control $u_2(t)$ can be established for every $(t, k(t))$ such that the system of ODE's

$$\left|\begin{array}{l} \dot{k}(s) = u(s) - \mu k(s), \\ \dot{u}(s) = \dfrac{1}{2a_3}[(\rho+\mu)(2a_3 u(s) + b_3) - 2a_1 k(s) - b_1] \end{array}\right. \tag{12}$$

starting from the point $k(t)$ and $u_2(t)$ for $s = 0$ has a solution satisfying $k(\infty) = \bar{k}_1$.

Then $k_{DNS} = k(t^*)$, where t^* is the intersection of the curves $\pi_1(t)$ and $\pi_2(t)$.

When we compare the values of Π_1 and Π_2 at the points $\bar{k}_1$, k^* and $\bar{k}_2$, the three inequalities in (10) may be:

(i) $\Pi_1(\bar{k}_1, \bar{u}_1) > \Pi_2(\bar{k}_1, \hat{u}_2)$, $\Pi_1(k^*, u_1^*) > \Pi_2(k^*, u_2^*)$ and $\Pi_1(\bar{k}_2, \hat{u}_1) < \Pi_2(\bar{k}_2, \bar{u}_2)$. Then there exists DNS point and it can be computed with the same calculations, formal swapping indexes 1 and 2.

(ii) $\Pi_1(\bar{k}_1, \bar{u}_1) < \Pi_2(\bar{k}_1, \hat{u}_2)$, $\Pi_1(k^*, u_1^*) < \Pi_2(k^*, u_2^*)$ and $\Pi_1(\bar{k}_2, \hat{u}_1) < \Pi_2(\bar{k}_2, \bar{u}_2)$. There is no DNS point in this case and we can replace $r(k)$ with $r_2(k)$.

4 Numerical example [3]

Let us consider the following values of coefficients for (3):

$$a_1 = a_2 = -0.1,\ b_1 = 5,\ b_2 = 10,\ c_2 = -110,\ a_3 = 1,\ b_3 = 1;$$

and the values

$$\rho = 0.1 \quad \text{and} \quad \mu = 0.2$$

for the dynamics (2).

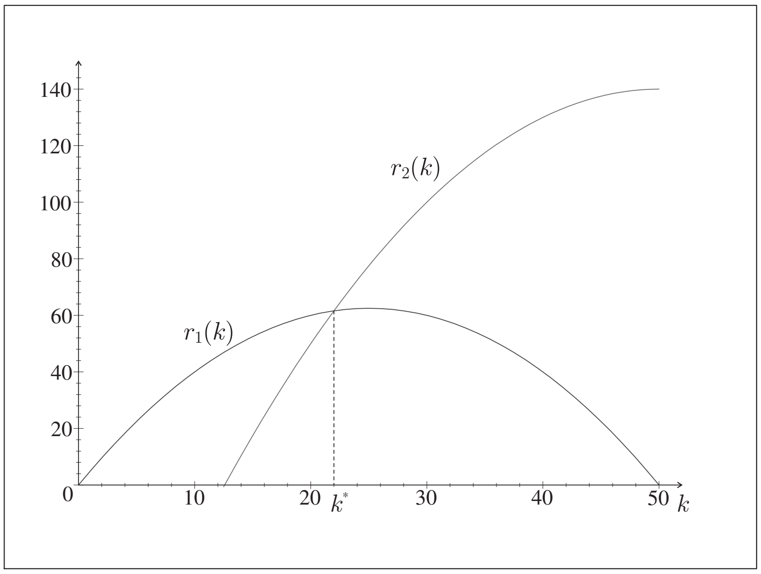

Figure 1: Graphs of $r_1(k)$ and $r_2(k)$

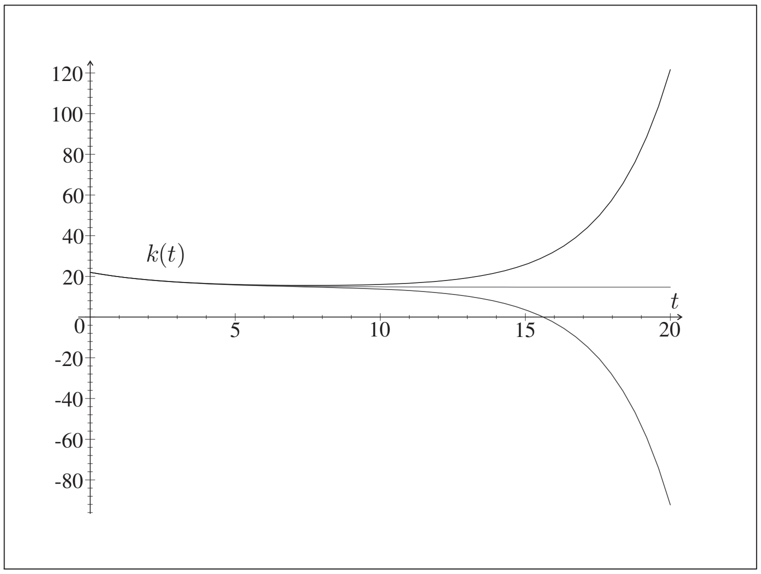

Figure 2: The trajectories $k(t)$ of (7) for $u(0)$ from (10) and $\delta = 0.01$ from

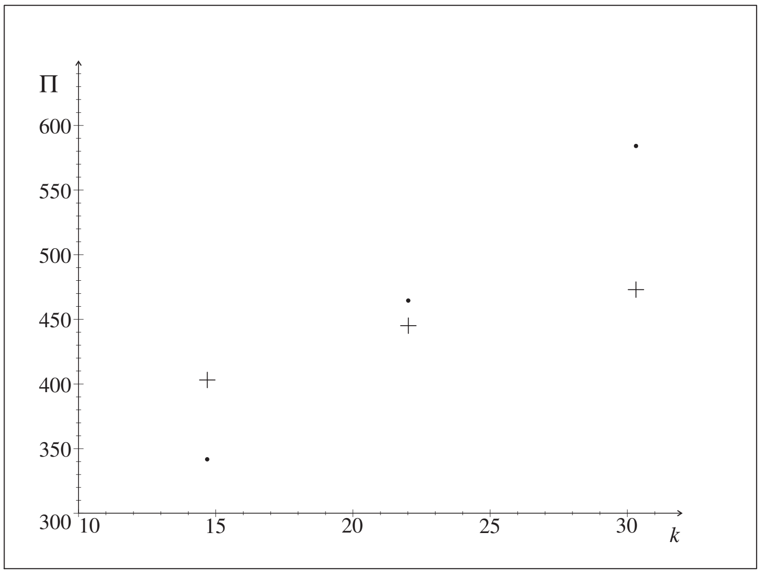

Figure 3: The optimal values (10) for $\bar{k}_1 = 14.69$, $k^* = 22$ and $\bar{k}_2 = 30.31$

We obtain $k^* = 22$ (Fig. 1) and $\bar{k}_1 = 14.69$, $\bar{k}_2 = 30.31$ from (8). Next we calculate $u_1^* = 1.82$, $u_2^* = 7.33$, $\hat{u}_1 = 0.54$ and $\hat{u}_2 = 8.45$. The following table and Fig. 3 presents (10).

k	u_1	$\Pi_1(k,u_1)$	u_2	$\Pi_2(k,u_2)$
$\bar{k}_1 = 14.69$	$\bar{u}_1 = 2.94$	403.0	$\hat{u}_2 = 8.45$	341.8
$k_{DNS} = 19.88$	2.14	434.6	5.81	434.6
$k* = 22.00$	$u_1* = 1.82$	445.0	$u_2* = 7.33$	464.6
$\bar{k}_2 = 30.31$	$\hat{u}_1 = 0.54$	473.0	$\bar{u}_2 = 6.06$	584.2

For the completeness the corresponding values at the DNS-point are also included in the table.

From (11) and (12) we calculate $\pi_1(t)$ and $\pi_2(t)$ on the time interval $[0, 1]$ (see Fig. 4). The intersection point is $k_{DNS} = 19.88$ for $t^* = 0.899$ and the optimal objective value is $\pi_1(t^*) = \pi_2(t^*) = 434.6$ (with precision 0.001). It follows that if we start with capital stock equal to $k_{DNS} = 19.88$, there are two possible scenario (investment plans) with one and the same optimal profit 434.6.

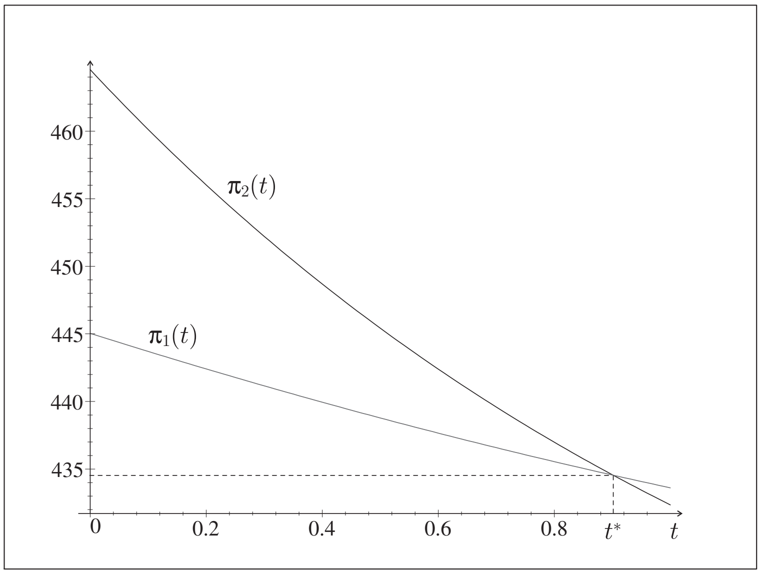

Figure 4: The objective functional values π_i

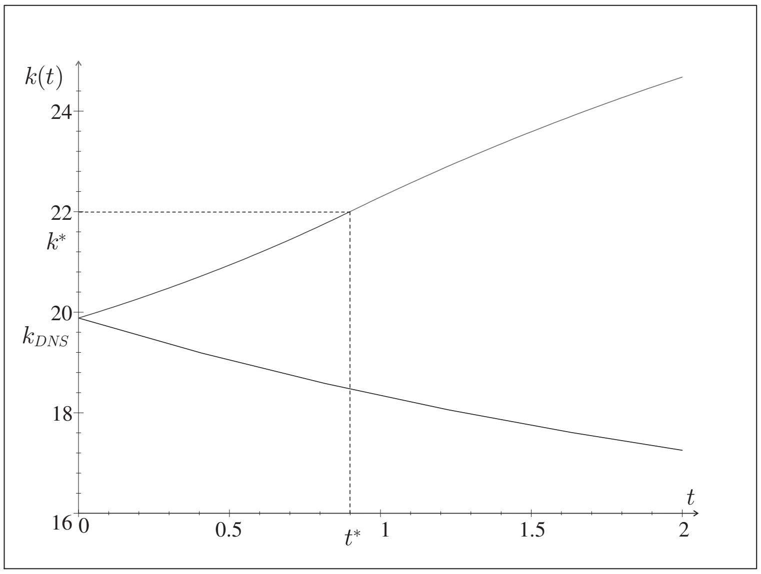

Figure 5: The capital stocks $k(t)$ for $k(0) = k_{DNS}$

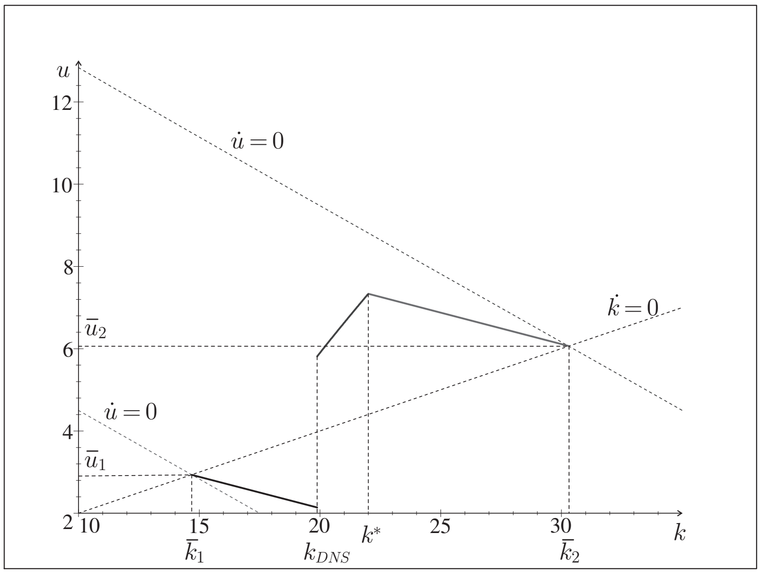

Figure 6: The phase portrait of optimal trajectories $(k(t), u(t))$ starting from $k(0) \in [\bar{k}_1, \bar{k}_2]$

(i) Leading to the long term steady state $(\bar{k}_1, \bar{u}_1) = (14.69, 2.94)$ with the initial $u_1(0) = 2.14$ and satisfying (7) with $i = 1$ (see Fig. 5, decreasing line).

(ii) Leading to the long term steady state $(\bar{k}_2, \bar{u}_2) = (30.31, 6.06)$ starting with initial $u_1(0) = 5.81$ according to (7) with $i = 1$. This works to $t = t^* = 0.899$, where we should switch to $i = 2$ and initial conditions $k(t^*) = k^* = 22$ and $u_2(t^*) = u_2^* = 7.33$ (see Fig. 5, increasing line).

The phase portrait of $(k(t), u(t))$ for initial capital stocks in $[\bar{k}_1, \bar{k}_2]$ is shown in Fig. 6. For any $k(0) \in [14.69, 30.31]$ the investment strategy $u(t)$ leads to one of the two steady states with an optimal profit.

5 Conclusions

The present article has investigated a simple cost-production model of a firm in the situation that the revenue function fails to be concave in some region of firm size. Only linear-quadratic model is investigated. However, the numerical method allows to establish the existence of DNS-point and to calculate it (if exists) for any "concave" revenue function $r(k)$ with a "convex kink” and any convex cost function $c(u)$.

References

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