Modelling Techniques for Uncertain Systems,
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Dynamic Interactive System for Analysis of Linear Differential Inclusions[1]

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1 Introduction

In the paper we present a numerical technique and its computer realization for approximation of the reachable set of a linear differential inclusion. In contrast to the other computer programs known to the authors and dealing with the same problem, our aim is to obtain approximation with guaranteed (and possibly controlled) accuracy. That is, we provide

– computer representable set that contains the reachable set (*guaranteed external approximation*);

– estimation of the Hausdorff distance between the approximation and the true reachable set (*guaranteed accuracy of approximation*);

– possibility the guaranteed accuracy of approximation to be prescribed (with certain bounds) by the user;

– appropriate tools for graphic representation of the approximation.

We first formulate the mathematical problem for which the software system `LINC` is implemented.

Let us consider the following **linear differential inclusion**

(1.1) $$\dot{x}(t) \in Ax(t) + BU, \quad t \in [0, T],$$

where X_0 and U are convex and compact subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$, $x(0) \in X_0$, and A, B are matrices with dimensions $n \times n$ and $n \times m$ respectively. We shall denote by $X_{[0,T]}$ the **trajectory bundle** of (1.1) on $[0, T]$, i.e.

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$$X_{[0,T]} = \{x(\cdot) : x(\cdot) \text{ is abs. continuous and satisfies (1.1) for a.e. } t \in [0,T]\},$$

and by $X(T)$ the **reachable set** of (1.1) at the time T, i.e.

$$X(T) = \{x(T) : x(\cdot) \in X_{[0,T]}\}.$$

For the solution of (1.1) we have:

$$X(T) = e^{AT} X_0 + \int_0^T F_T(t) \mathrm{d}t, \text{ where} \tag{1.2}$$

$$F_T(t) = \{e^{A(T-t)} Bu, u \in U\}. \tag{1.3}$$

Thus the problem is to constructively approximate the above integral. To do that we employ the standard approach of calculation of the support function of $X(T)$, involving discretization of the integral. However, in order to be able to efficiently control the error of discretization without redundantly decreasing the step length we make use of the recently developed second order discretization techniques, briefly described below.

Let us consider the Lipshitz multivalued function $F(\cdot)$: $[0,T] \Longrightarrow \mathbf{R}^n$ with convex and compact values. By $\rho(\cdot, F(t))$ we denote the **support function** of the set $F(t)$. Clearly this function is Lipshitz and differentiable a.e. with respect to t. Suppose that there exists a constant $W > 0$ such that

$$\mathrm{Var}\ \frac{\partial}{\partial t} \rho(l, F(t)) \leq W \tag{1.4}$$

for every $l \in \mathbf{R}^n, ||l|| = 1$, where Var denotes the variation on the interval $[0,T]$.

Let N be positive integer, $h = T/N$ and $t_k = kh$, $k = 0,1,\ldots,N$. An approximation scheme of order α can be given by

$$G_N = \mathrm{co}\left\{ h \sum_{k=0}^{N-1} \sum_{i=1}^{m} a_i F(t_k + \tau_i h) \right\}, \tag{1.5}$$

where $0 \leq \tau_1 < \ldots < \tau_m \leq 1$ and $a_1, \ldots, a_m \geq 0$ are chosen in such way that

$$\int_0^1 P(t) \mathrm{d}t = \sum_{i=1}^{m} a_i P(\tau_i)$$

for every polynomial P of degree less than α.

Proposition 1.1 (Veliov [2]) *Let $F(\cdot)$: $[0,T] \Longrightarrow \mathbf{R}^n$ be a Lipshitz multi-valued function with convex and compact values, G_N be an approximation scheme of order* 1 *and assumption* (1.4) *is fulfilled. Then*

$$H\left(\int_0^T F(t)\mathrm{d}t, G_N\right) \leq T^2\left(1+\sum_{i=1}^m a_i\right) W/N^2,$$

where H is the Hausdorff distance in $\mathbf{R}^n$.

Moreover, condition (1.4) is ever fulfilled for (1.2), (1.3) ([2], Proposition 2). Thus the set-valued version (1.5) of linear quadrature formulae of order 1 provides approximation of second order accuracy with respect to the step h for the multivalued integral (1.2). It is remarkable that approximations of such type with higher order of accuracy are shown not to exist, in general.

2 Algorithm

In our algorithm we apply Proposition 1 with $m = 1$, $a_1 = 1$ and $\tau_1 = 0.5$. The error $E_1(h)$ of this approximation scheme can be estimated by the expression:

$$E_1(h) \leq 2h^2(2+5T\,||A||)\,||A||\,||\mathrm{e}^{AT}||\,||B||\,||U||, \tag{2.1}$$

where $||U|| = \max\{||u||\colon u \in U\}$.

For the computer realization we need to calculate the exponent of the matrix A. There are different methods for solving this problem. We apply the approach based on the solution of the corresponding differential equation. The Banach fixed point theorem is used for obtaining a priori bounds for the solution of the equation

$$\dot{x} = Ax, \quad x(0) \in X_0,$$

where X_0 is a compact convex set in $\mathbf{R}^n$. Let $R(t)$ denote the set containing all points $x(t)$, where $x(\cdot)$ is a solution of the equation with starting point $x(0) \in X_0$. Then the following lemma is true:

Lemma 2.1 *Let Y be a compact convex subset in $\mathbf{R}^n$ and $h > 0$ be a positive real such that the following relations are true:*

$$h||A|| < 1, \quad X_0 \subset Y \text{ and } X_0 + h\,AY \subset Y.$$

Then $R(h) \subset Y$.

A similar idea is used for obtaining a priori bounds for the solutions of ordinary differential equations by R. Lohner [1]. The approximation scheme error E_2 can be estimated by the expression:

$$E_2 \leq \text{diam}(\mathcal{A}(A,T))(\|X_0\| + T\|B\|\,\|U\|),$$

where diam$(\mathcal{A}(A,T))$ denotes the diameter of the box $\mathcal{A}(A,T))$ which approximates $e^A T$. This expression shows that the error E_2 can be done arbitrarily small making diam$(\mathcal{A}(A,T))$ sufficiently small.

We suppose that the support functions $\rho(\cdot, X_0)$ and $\rho(\cdot, U)$ are known. In addition we suppose extreme points $x_0 \in X_0$ and $u_0 \in U$ with the properties $\rho(e, X_0) = \langle e, x_0\rangle$ and $\rho(e, U) = \langle e, u_0\rangle$ can be found for every direction e. Then we can calculate the support function $\rho(\cdot, G_N)$ and extreme points $x(e) \in G_N$ such that $\rho(e, G_N) = \langle e, x(e)\rangle$ for every direction e. If $E_1 + E_2 < |\rho(e, G_n) - \rho(-e, G_n)|$ then convex hull of points $x(e) - (E_1 + E_2)e$ is a internal polygonal approximation of the reachable set. An external polygonal approximation is the intersection of half spaces, determined by support hyperplanes for the set $G_N + \{y \in \mathbf{R}^n : \|y\| < E_1 + E_2\}$ at the points $x(e) + (E_1 + E_2)e$.

The accuracy of the approximation depends on two parameters: the step length h and the number of directions e for calculation of the support function. The dependence of h can be a priori estimated by (2.1), while the influence of the second parameter on the accuracy depends on the geometry of the reachable set and is being evaluated during the computation of the latter.

3 Implementation

Here we briefly outline dynamic interactive system `LINC` which realize the approach described above. This system is written in `TURBO PASCAL 5.0` – Borland Int. Inc. for microcomputer `IBM PC` under `MS DOS`. The main features of the system are:

– calculation of one, two and three dimensional projections of an approximation of the reachable sets;

– calculation of the extreme points of the reachable sets on directions obtained by discretization of the n-dimensional sphere;

– facilities for visualization of one, two and three dimensional projections of the reachable set on different subspaces;

– visualization of the dynamics of one and two dimensional projections.

4 Examples

Example 1. Let us consider a control system

$$\begin{aligned} \dot{x} &= -x - y + u, \\ \dot{y} &= 2x - 0.4y - u, \end{aligned}$$

$$X_0 = \{(x, y)\colon -1 \le x \le 1, -1 \le y \le 1\}, \quad U = [-0.2, 2].$$

On Fig. 1 the reachable sets at different moments are given. The step and corresponding errors can be seen in the following table:

T	h	E_1	E_2
0.5	0.01	0.012	0.00038
1	0.01	0.021	0.00081
2	0.01	0.040	0.00015

On Fig. 2 and Fig. 3 the trajectory bundle and its one-dimensional projection on the line $y = 0$ are drawn.

Example 2. Let us consider a control system

$$\begin{aligned} \dot{x} &= -x - y + u, \\ \dot{y} &= 2x - 0.4y - 2z, \\ \dot{z} &= y - 0.1z, \end{aligned}$$

X_0 is the 3-dimensional unit ball and $U = [-1, 1]$.

On Fig. 4 3-dimensional reachable set at the moment $T = 1$ is given. The step is $h = 0.01$ and the accuracy $E_1 = 0.032$ and $E_2 = 0.00047$.

Also approximations of the reachable sets are obtained for some examples up to dimension 10 for the phase variable and up to dimension 2 for the control.

References

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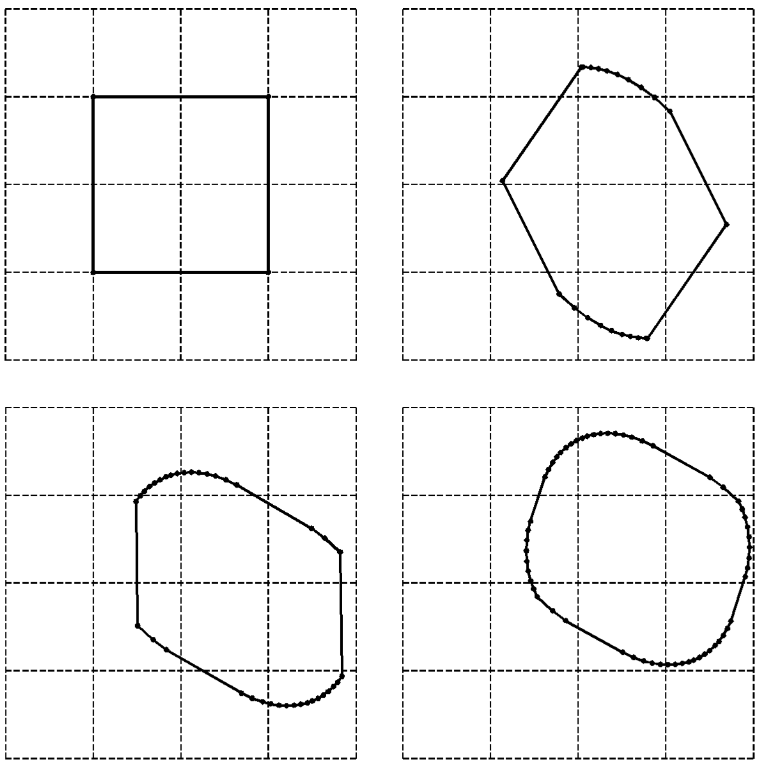

Fig. 1. Reachable sets of Example 1

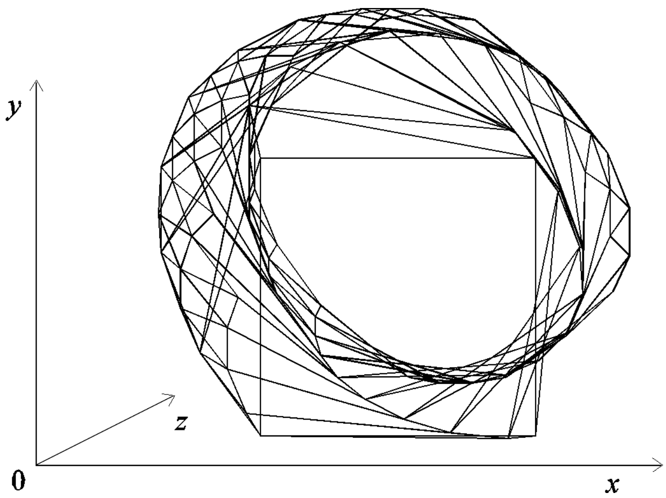

Fig. 2. Trajectory bundle of Example 1 for $T \in [0, 2]$

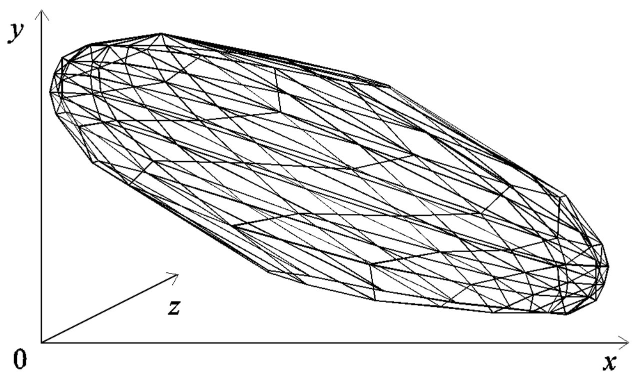

Fig. 4. Reachable set of Example 2 at $T = 0.5$