Chaotic Attractor Reconstruction and Applications to Variable Stars

Nikolai K. Kirov[1]
Institute of Mathematics, Bulgarian Academy of Sciences
Akad. G. Bontchev St., bl.8, 1113 Sofia, Bulgaria

Abstract

A short mathematical background of fractal dimensions, the correlation integral method for computing spectrum of dimensions and attractor reconstruction are presented. A corresponding software system for IBM-PC is created and some of its possibilities are described. The system is used for calculating the spectrum of dimensions of fractal sets, discrete dynamical systems and attractors of differential equations. Also, the results of processing real electrophotometric data for the cataclysmic variables TT Ari and KR Aur are obtained and discussed.

1 Introduction

There are different methods for finding the dimension of the attractors. The correlation integral method (introduced by Grassberger and Procaccia [6]) is one of the widespread methods for practical computations. This method can be applied in astronomy for objects possessing any observed (and measured) variability (see Kolláth and Nuspl [9]). Such investigations aim at:

- determining low-dimensional attractor global evolution of the system which can be described by non-linear differential equations showing deterministic chaos in their behaviour (cf. Auvergne and Baglin [2], Atmanspacher *et al.* [1], Harding *et al.* [7]);
- determining local scaling properties (white or shot noise); a system consisting of a number of uncorrelated elements which appear at random and live only a short time (cf. Cannizzo and Goodings [5], Lehto *et al.* [11]);
- comparing similar variable astronomical objects (galaxies, variable stars, *etc.*) in order to do its classification (cf. Lehto *et al.* [11]).

The main problems in the determination of correlation dimension are high noise level of the observations and the small length of the data (see Lochner *et al.* [12], Norris and Matilski [13]). In this paper, we apply the correlation integral method for a number of simulated light curves (including noise) and real data. Comparison analysis of the graphics obtained is the main tool in our investigations.

2 Generalized attractor dimensions

Balatoni and Renyi [3] and then Grassberger and Procaccia [6] introduced the notion of *attractor dimensions*. Let A be an attractor (limit attracting set) of a dynamical system. Cover A with volume elements (spheres, cubes, *etc.*) each with diameter ε and let $N(\varepsilon)$ be the minimum number

Proceedings of 5^{th} ESO/ST-ECF Data Analysis Workshop, April 26-27, 1993
[1]This work was partially supported by grant No 58/91 of the Bulgarian Ministry of Science and Higher Education.

of such volume elements needed to cover A. *Generalized dimension of order* q (also spectrum of dimensions or Renyi dimensions) is introduced by

$$D^{(q)} = \lim_{\varepsilon\to 0} \frac{\log M^{(q)}(\varepsilon)}{\log \varepsilon}, \quad M^{(q)}(\varepsilon) = \left(\sum_{i=1}^{N(\varepsilon)} p_i{}^q \right)^{\frac{1}{q-1}}, \tag{1}$$

where p_i is the relative frequency with which a typical trajectory (forming the attractor) enters the i-th volume element of the covering. Usually, a dimension of order 0 is called *capacity*, *Hausdorff dimension* or *fractal dimension*, $D^{(1)}$ – *information dimension* and $D^{(2)}$ – *correlation dimension.*

$$D^{(0)} = \lim_{\varepsilon\to 0} \frac{-\log N(\varepsilon)}{\log \varepsilon}, \quad D^{(1)} = \lim_{\varepsilon\to 0} \frac{\sum_{i=1}^{N(\varepsilon)} p_i \log p_i.}{\log \varepsilon}, \quad D^{(2)} = \lim_{\varepsilon\to 0} \frac{\log \sum_{i=1}^{N(\varepsilon)} p_i{}^2}{\log \varepsilon}. \tag{2}$$

Let us assume that a trajectory $\{x_i\}_{i=1}^{\infty}$ forms the attractor A. For a given $\varepsilon > 0$, generalized correlation function of order q is introduced by

$$C^{(q)}(\varepsilon) = \lim_{N\to\infty} \left(\frac{1}{N} \sum_{i=1}^{N} \left(\frac{1}{N} \sum_{j=1}^{N} \theta(\varepsilon - \|x_i - x_j\|) \right)^{q-1} \right)^{\frac{1}{q-1}}, \tag{3}$$

where x_i, x_j are trajectory points, θ is Heaviside function, *i.e.* $\theta(z) = 1$ for $z > 0$ and $\theta(z) = 0$ for $z \leq 0$. Then the correlation integral of order q is given by the following formula

$$C^{(q)} = \lim_{\varepsilon\to 0} \frac{\log C^{(q)}(\varepsilon)}{\log \varepsilon}. \tag{4}$$

3 Reconstruction of attractors

A remarkable result, first shown by Takens [16], allows a strange attractor to be reconstructed from a sampled time waveform of just one component of the state. For a system where one or more of the state variables cannot be measured directly, reconstruction may be the only way to observe the attractor. This is the usual case in astronomy.

Let $\{x_k\}_{k=0}^{N}$ be a sequence of measurements of an attractor, contained in K-dimensional manifold. Then we build sequences of d-dimensional points $X_i = (x_i, x_{i+1}, \ldots, x_{i+d-1})$, $i = 1, 2, \ldots, N-d$ and calculate the dimension in the space R^d. According to Takens' theorem, the dimension will not depend on d when $d \geq 2K+1$. The correlation integral method consists of first calculating the correlation function (Equation 3) in d-dimensional space and then determining the slope of the linear part of correlation function $C^{(q)}(\varepsilon)$ vs. ε.

4 Methods and program system

A program system is created for calculating the correlation integral $C^{(q)}$. It works on IBM-PC (386 and better) and supports EGA and VGA graphics display. The first (computational) part of the system calculates the generalized correlation function $C^{(q)}(\varepsilon)$ for sequences of values for ε and q, and for embedding dimensions $d = 1, 2, \ldots, d_{max}$. Our main efforts have been directed to an efficient realization of the correlation integral method (cf. Parker and Chua [14]). Data up to 3000 points can be processed into a reasonable time interval.

The second part represents the results (the calculated correlation functions) as different graphics. It provides means for interactive work with the user when drawing basic graphics of the following kind:

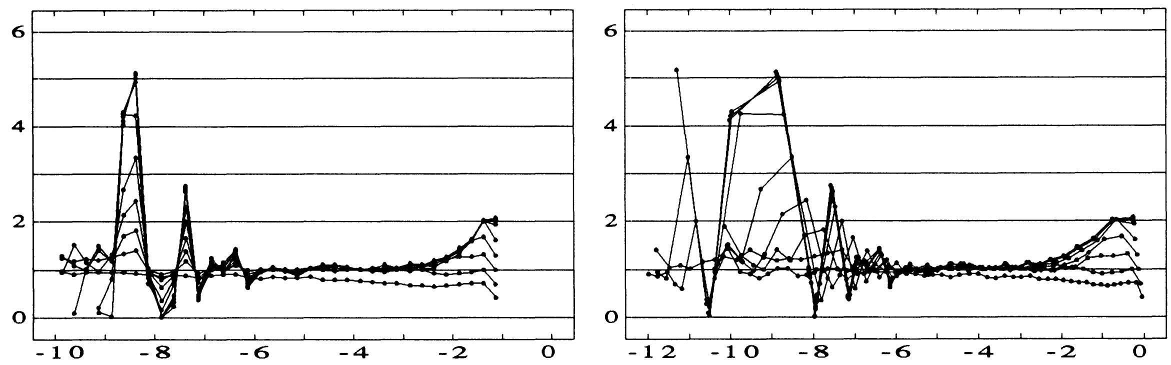

Figure 1: The correlation function of the attracting limit cycle for Van der Pol's equation ($N = 1000, d = 1 \dots 10$) – slope vs. $\log \varepsilon$ and slope vs. $\log C^{(2)}(\varepsilon)$.

- artificial 2-dimensional phase space x_k vs. $x_{(k+l)}$ for different l and for different subsequences of the data sequence x_k;
- the correlation integral of order q, $C^{(q)}$, as a logarithmic graphic $\log C^{(q)}(\varepsilon)$ vs. $\log \varepsilon$ for a single embedding dimension d or for a set of embedding dimensions;
- the slope of the correlation function of order q vs. $\log \varepsilon$ and vs. $\log C^{(q)}(\varepsilon)$, again for a single embedding dimension d or for a set of embedding dimensions;
- the correlation integral of order q, $C^{(q)}$ vs. q and vs. embedding dimensions d.

Here, we include only calculations of the correlation function of order 2 $\log C^{(2)}(\varepsilon)$, as a first step of our investigations. The slope in these cases can be clearly seen when drawing graphics slope vs. ε and slope vs. $C^{(q)}(\varepsilon)$.

5 Results

5.1 Simulated light curves

Some simulated light curves produced from deterministic functions are used as test examples for the correlation integral method and for our software system. The dimensions of a few attracting sets of the discrete dynamical systems Logistic and Henon maps (for various values of the parameters) have been calculated. Also, the dimension of the Serpinski triangle fractal set has been obtained using a random algorithm for building this set (see Barnsley [4]). Data from some trajectories of differential equations (represented attractind sets) have been processed.

Van der Pol's equation

$$\dot{x} = y, \quad \dot{y} = (1 - x^2)y - x \tag{5}$$

is a classical example of an attracting limit cycle. This attractor is 1-dimensional, of course (see Figure 1). The effect of adding 10 percent white noise is seen in Figure 2. Our experimental data consist of x-measurements, obtained by numerically calculated limit cycle using Runge-Kutta scheme of order 2.

The Lorenz attractor is produced by one of the simplest sets of differential equations demonstrating chaotic behaviour:

$$\dot{x} = -\sigma x + \sigma y \quad \dot{y} = -xz + rx - y \quad \dot{z} = xy - bz, \tag{6}$$

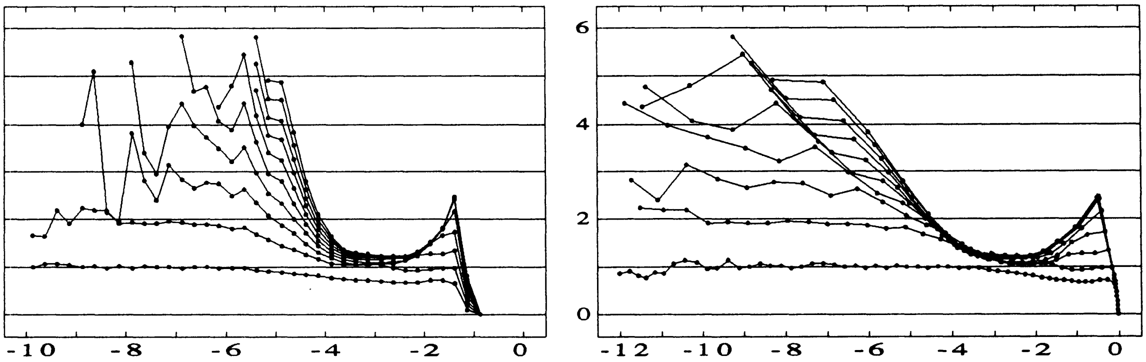

Figure 2: The correlation function of the attracting limit cycle for Van der Pol's equation plus 10 percent noise – slope vs. $\log \varepsilon$ and slope vs. $\log C^{(2)}(\varepsilon)$.

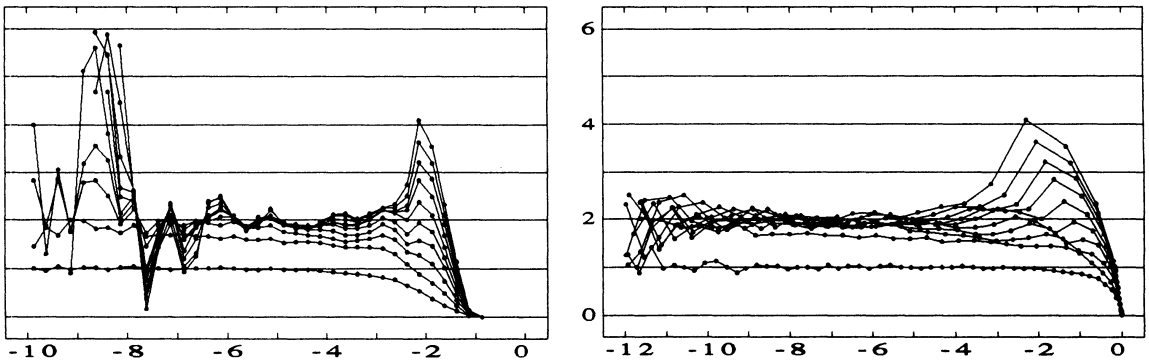

Figure 3: The correlation function of the Lorenz light curve ($N = 1200, d = 1 \ldots 10$) – slope vs. $\log \varepsilon$ and slope vs. $\log C^{(2)}(\varepsilon)$.

where $\sigma = 10, r = 28, b = 8/3$. The dimension of the attractor is approximately 2.05 (Grassberger and Procaccia [6]). An Euler numerical scheme is applied for chaotic trajectory calculation with step 0.02. More detailed investigation of this attractor is given by Lehto *et al.* [11]. Our results are given in Figures 3 and 4.

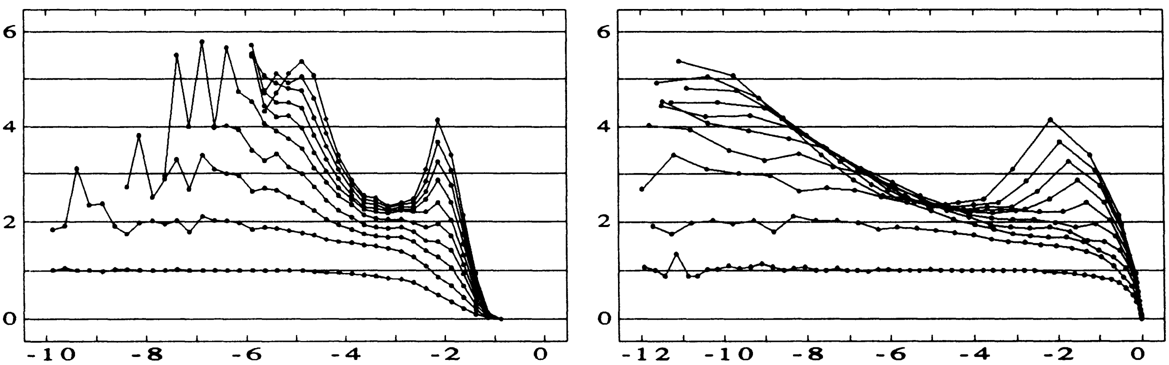

Figure 4: The correlation function of the Lorenz light curve with 10 percent noise – slope vs. $\log \varepsilon$ and slope vs. $\log C^{(2)}(\varepsilon)$.

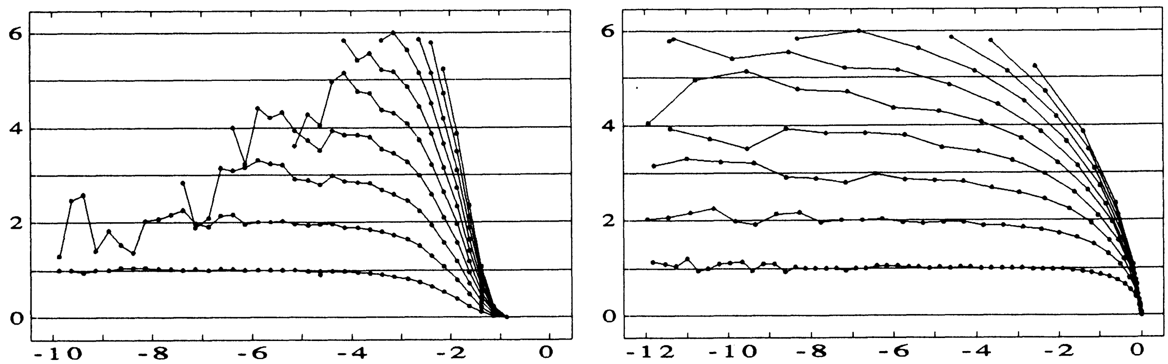

Figure 5: The correlation function for Gaussian distributed noise ($\sigma = 1, N = 1200, d = 1 \ldots 10$) – slope vs. $\log \varepsilon$ and slope vs. $\log C^{(2)}(\varepsilon)$.

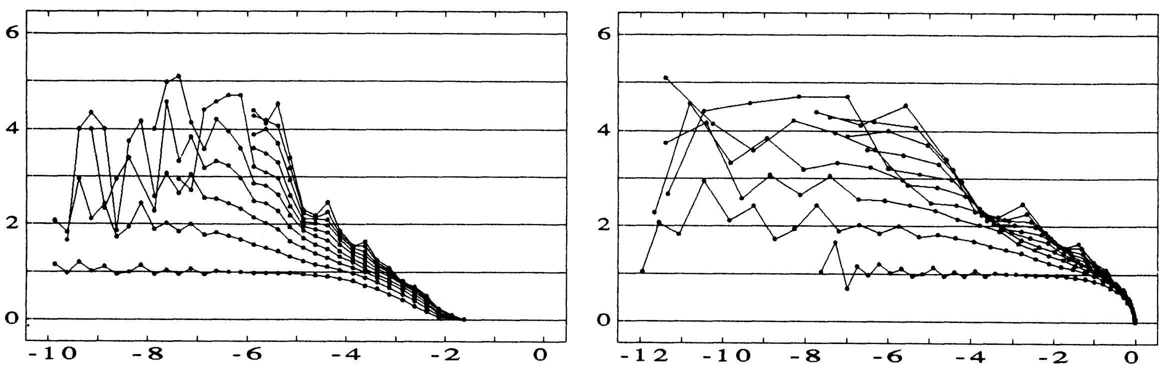

Figure 6: The correlation function for TT Ari light curve ($N = 1050, d = 1 \ldots 10$) – slope vs. $\log \varepsilon$ and slope vs. $\log C^{(2)}(\varepsilon)$.

5.2 White noise.

The dimension of the ideal white noise (random quantity, uniformly or normally distributed) in R^d is equal to the embedding dimension d. In practice, as Figure 5 shows, the correlation function has a slope which increases with the higher embedding dimension.

5.3 TT Arietis and KR Aurigae.

Processing data of these cataclysmic stars were obtained by Kraicheva *et al.* [10] for TT Arietis, and for KR Aurigae by Popov and Antov [15].

The observations were made in U-color of the standard UBV system using a single channel photon-counting photoelectric photometer, attached to 60 cm Cassegrain telescope in the National Astronomical Observatory Rozhen. Photometric data reduction has been made by the program system APR (Kirov *et al.* [8]).

A definite assertion for existence (or non existence) of a low-dimensional attractor can not be given on the basis of the presented graphical results (Figures 6 and 7). Probably, the further more detailed investigations, using the full possibilities of the software system can lead up to a definite answer. Also, Fourier techniques to search for and remove any strong periodic components will be used and some predictive procedures can help us to establish the presence of deterministic chaos.

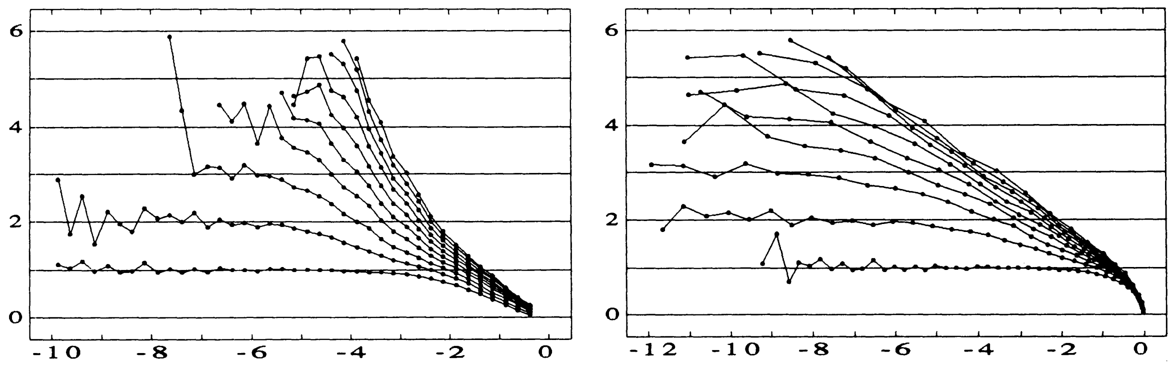

Figure 7: The correlation function for KR Aur light curve ($N = 1200, d = 1 \ldots 10$) – slope vs. $\log \varepsilon$ and slope vs. $\log C^{(2)}(\varepsilon)$.

References

[1] Atmanspacher, H., Scheingraber H., Voges, V.: 1988, *Phys. Rev. A,* **37**, 1314.

[2] Auvergne, M., Baglin A.: 1986, *Astr. Astrophys.,* **168**, 118.

[3] Balatoni, J., Renyi, A.: 1976, *Selected Papers of A. Renyi*, ed. P. Turan (Akad. K., Budapest), **Vol.1**, 588.

[4] Barnsley, M.: 1988, *Fractals Everywere*, Acad. Press, San Diego.

[5] Cannizzo, J.K., Goodings D.A.: 1988, *Ap. J.,* **334**, L31.

[6] Grassberger, P., Procaccia, I.: 1983, *Phys. Rev. Lett.,* **50**, 448.

[7] Harding, A.K., Shinbrot, T., Cordes, J.M.: 1990, *Ap. J.,* **353**, 588.

[8] Kirov, N.K., Antov, A.P., Genkov, V.V.: 1991, *Compt. rend. Acad. bulg. Sci.*, **44**, 11, 5.

[9] Kolláth, Z., Nuspl, J.: 1990, *Rapid variability of OB-stars: Nature and diagnostic value, Proc. ESO Workshop*, ed. D. Baade, 363.

[10] Kraicheva, Z., Antov, A., Genkov, V.: 1989, *Commun. Konkoly Obs. Hung. Acad. Sci.*, **10**, 93, 243.

[11] Lehto, H.J., Czerny, B., M^{C}Hardy I.M.: 1993, *Mon. Not. R. Astron. Soc.,* **261**, 125.

[12] Lochner, J.C., Swank, J.H., Szymkowiak, A.E.: 1989, *Ap. J.*, **337**, 823.

[13] Norris, J.P., Matilski, T.A.: 1989, *Ap. J.*, **346**, 912.

[14] Parker, T.S., Chua, L.O.: 1989, *Practical Numerical Algorithms for Chaotic Systems*, Springer, New York.

[15] Popov, V., Antov, A.: 1989, *Comm. 27 IAU Inf. Bull. Var. Stars*, N3307.

[16] Takens, F.: 1981, *Dynamical systems and Turbolence*, ed. Rand, D.A. etc., Springer, Berlin, 336.